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(XI & XII)

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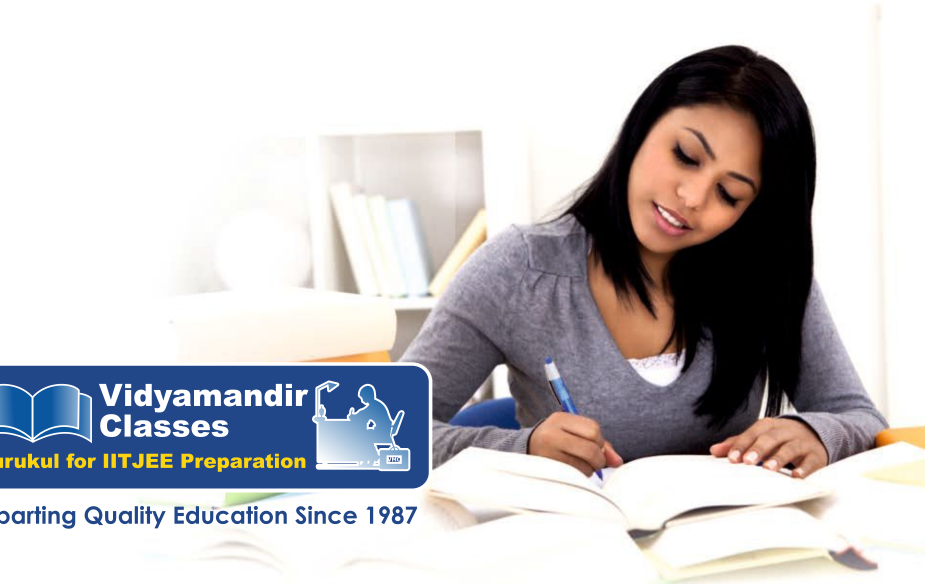
MATHS MUSING
10 GREAT PROBLEMS



MATHEMATICS today

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Corporate Office:
Plot 99, Sector 44 Institutional Area,
Gurgaon -122 003 (HR), Tel : 0124-6601200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office:
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat

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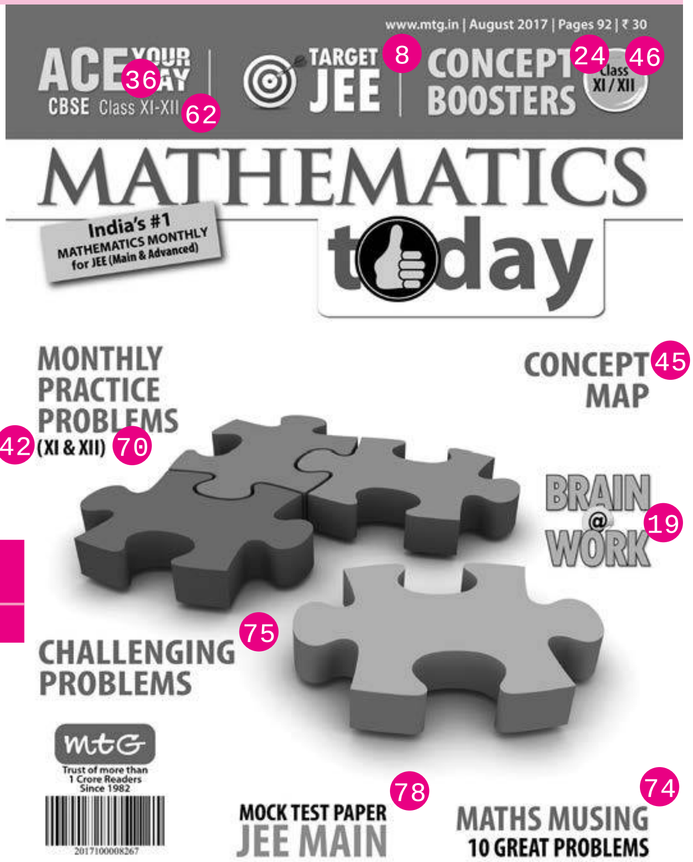

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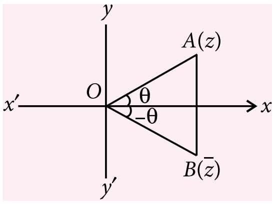
Editor : Anil Ahlawat

TARGET JEE

Complex Numbers

Conjugate complex number

If $z = a + ib$ be a complex number, then conjugate of z is denoted by $\bar{z}$ and defined by $\bar{z} = a - ib$, where $\bar{z}$ is the mirror image of z about real axis in argand plane.



Remark :

- $z + \bar{z} = 2\,\text{Re}(z) = 2a$
- $z - \bar{z} = 2i\ \text{Im}(z) = 2ib$
- $\text{Arg}(z) = -\text{Arg}(\bar{z})$
- $|z|^2 = |\bar{z}|^2 = z\bar{z} = a^2 + b^2$ (purely real number)
- If z lies in the first quadrant, then $\bar{z}$ lies in the 4th quadrant.

Trigonometric / Polar form of a complex number

Let $z = a + ib$, be a complex number

Put $a = r\cos\alpha$, $b = r\sin\alpha$

$\therefore$ $|z| = r$ and $\arg(z) = \alpha$

$\therefore$ $z = r(\cos\alpha + i\sin\alpha)$ and $\bar{z} = r(\cos\alpha - i\sin\alpha)$ are polar form of z and $\bar{z}$ respectively.

Remark :

$\cos\alpha + i\sin\alpha$ is also written as $\text{cis}\alpha$.

Euler form of $z = a + ib$ is given by $e^{i\alpha}$

Cube roots of unity

If x be the cube root of 1, *i.e.* $x = (1)^{1/3}$, then

$$x = 1, \frac{-1 - i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2} \quad \text{or} \quad 1, \omega, \omega^2 \quad \text{where}$$

$$\omega = \frac{-1 - i\sqrt{3}}{2},\ \omega^2 = \frac{-1 + i\sqrt{3}}{2}$$

- The polar form of 1, ω and ω^2 are $\cos 0° + i\sin 0°$, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$
- The euler (exponential) form of 1, ω, ω^2 are given by $e^0, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$
- The three cube roots of unity when plotted on the argand plane constitutes the vertices of an equilateral triangle.
- Complex cube roots of unity are square conjugate and reciprocal of each other.
- $1 + \omega + \omega^2 = 0$.

  $$1 + \omega^r + \omega^{2r} = \begin{cases} 0, & \text{if } r \text{ is not a multiple of } 3 \\ 3, & \text{if } r \text{ is a multiple of } 3 \end{cases}$$
- If $a, b, c \in R$ and ω is cube root of unity, then

  $a^3 \pm b^3 = (a \pm b)(a \pm \omega b)(a \pm \omega^2 b)$;

  $a^2 \pm ab + b^2 = (a \mp b\omega)(a \mp b\omega^2)$;

  $a^2 + b^2 + c^2 - ab - bc - ca$
  $= (a + \omega b + c\omega^2)(a + b\omega^2 + c\omega)$;

  $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)$
  $(a + \omega^2 b + \omega c)$

n^{th} roots of unity

If $1, \alpha_1, \alpha_2, ..., \alpha_{n-1}$ are n, n^{th} roots of unity, then

- They form a G.P. with common ratio $(r) = e^{i\left(\frac{2\pi}{n}\right)}$
- $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p$
  $$= \begin{cases} 0, & \text{if } p \text{ is not an integral multiple of } n \\ n, & \text{if } p \text{ is an integral multiple of } n \end{cases}$$
- $$1 \cdot \alpha_1 \cdot \alpha_2 \cdot .... \cdot \alpha_{n-1} = \begin{cases} 1, & \text{if } n \text{ is odd} \\ -1, & \text{if } n \text{ is even} \end{cases}$$
- $$(1 + \alpha_1)(1 + \alpha_2)....(1 + \alpha_{n-1}) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

By : **R. K. Tyagi,** Retd. Principal, HOD Maths, Samarth Shiksha Samiti, New Delhi

Best of Luck

for your Journey Ahead

JEE Achievers - 2017

Keshav Gupta

MIT, USA(2017-21)
JEE Adv. AIR-269
JEE Main AIR-232

I am Keshav Gupta, student at KCS Educate during the sessions 2015-16 and 2016-17. The teachers at KCS Educate constantly motivated me and guided me towards the target. In the classroom, the focus always remained on keeping the students interested in the topics. Apart from academics, great emphasis was layed on motivational support for students. It has truly been a catalyst for success.

Keshav
Keshav Gupta

Tushar Agrawal

JEE Adv. AIR-609
JEE Main C.G. Topper
KVPY Qualified

I am Tushar Agrawal. I joined KCS for Two year program. The lectures delievered by Sir were very interacting and motivating. It never felt to me that "I need to study" but like a pleasure flow and eventually I began to enjoy studies. Further the tests and assignments helped me to understand my weak areas and my speed. Avnish Sir motivated at each stage and was ready to help at any time. I want to thank Avnish Sir and his team for their support.

Tushar

BHILAI OFFICE 157, New Civic Centre, Bhilai, Dist. Durg (C.G.)
Telephone : **0788-6454505**

DELHI OFFICE 1B, Block GG1, Vikaspuri, New Delhi - 110018

CIN No. U74140DL2011PTC227887

info@kcseducate.in | www.kcseducate.in | facebook.com/kcseducate

More about the n^{th} roots of unity

- Consider the polynomial equation $z^n = 1$, the roots of the equation are n numbers and these n numbers are known as n^{th} roots of unity.
  To determine the roots of $z^n = 1$, writing the polar form of unity.
  $\therefore \quad z^n = 1 = \cos 0° + i\sin 0°$ and after generalize it we write
  $$z^n = \cos 2k\pi + i\sin 2k\pi,\ k \in I$$
  $$\Rightarrow \quad z = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$
  $$\Rightarrow \quad z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{i\frac{2k\pi}{n}} \quad \text{(Euler form)},\ k \in I$$
  If we assign different integral values to k then we always get the roots of $z^n = 1$ out of which some of them may be repeated. These are known as n, n^{th} roots of unity given by $e^{i\frac{2k\pi}{n}}$, $k = 0, 1, 2, \ldots (n-1)$.
  If we take $\alpha = e^{i\frac{2\pi}{n}}$ then n, n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.
- If these n, n^{th} roots of unity are represented on the argand plane, then these roots form the vertices of a regular n-gon whose centre lies at origin and radius of circumcircle of the n-gon is $\left|\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right|$ which is equal to 1. Thus we have
  $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$
  $\therefore \quad z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2) \ldots (z-\alpha^{n-1})$

Equation of straight line in complex form

- The equation of a line passing through two given points z_1 and z_2 is given by
  $$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$
  $$\text{or} \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$
  which is the condition for three complex numbers z, z_1, z_2 to be collinear.
- The general equation of a line is given by $\bar{a}z + a\bar{z} + b = 0$, where a is complex number and b is real number. The slope of this line is given by $-\frac{\text{Re}(a)}{\text{Im}(a)}$, whose distance from a point z_1 is given by $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$.
- The complex slope of line making angle α with real axis is $\omega = e^{2i\alpha}$.
- If a line passes through the point z_1 and z_2 then its complex slope is given by $\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$. The real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $-i\left(\frac{a+\bar{a}}{a-\bar{a}}\right)$ and complex slope is $-\frac{a}{\bar{a}}$ *i.e.* $-\frac{\text{coefficient of } \bar{z}}{\text{coefficient of } z}$ where $b \in R$ and $a \in Z$, a complex number.
- The parametric form of equation of line through two points z_1 and z_2 is given by $z = z_2 + \alpha(z_1 - z_2)$, $\alpha \in R$ (known as parameter).
- $|z - a| = |z - b|$ is the equation of perpendicular bisector of the line joining the points a and b.
- If $a, b, c \in R$ such that $az_1 + bz_2 + cz_3 = 0$ where $a + b + c = 0$ and a, b, c are not all simultaneously zero, then z_1, z_2, z_3 are collinear.

Triangles and circles in terms of complex number

- If the vertices of a triangle ABC are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the lengths of sides opposite to the vertices A, B and C respectively, then
  - centroid of $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$
  - incentre of $\Delta ABC = \frac{az_1 + bz_2 + cz_3}{a+b+c}$
  - circumcentre of ΔABC
    $$= \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$
  - orthocentre of ΔABC
    $$= \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$
    $$\text{or} \quad \frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$$
- Area of Δ formed by the points $A(z_1), B(z_2), C(z_3)$ is given by $\frac{1}{4}\left\|\begin{matrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{matrix}\right\|$
- If $A(z_1)$, $B(z_2)$ where $z_1 \neq z_2 \neq 0$ are two complex numbers in the argand plane such that $|z_1 + z_2| = |z_1 - z_2|$ and O be the origin, then
  (1) $\arg(z_1/z_2) = \pm\, \pi/2$
  (2) circumcentre of $\Delta AOB = \frac{z_1 + z_2}{2}$
  (3) orthocentre of ΔAOB lies at origin.
- The equation of circle centred at z_0 and radius R is given by $|z - z_0| = R$ or $|z - z_0|^2 = R^2$
  $$\Rightarrow \quad z\bar{z} - \bar{z}_0 z - \bar{z}z_0 + z_0\bar{z}_0 - R^2 = 0$$
  which is of the form $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$, where a

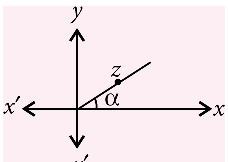
is a complex number and b is real number. The centre of the circle is $-a$ and radius $\sqrt{|a|^2-b}$, the circle will be real if $a\bar{a}-b\geq 0$.

- The equation of the circle described on the line segment joining z_1 and z_2 as diameter is given by
$$(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$$
or $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\pm\dfrac{\pi}{2}$
- $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda,\ \lambda>0,\ \lambda\neq 1$ represent a circle having diameter AB when A and B divided the join of z_1 and z_2 in $\lambda:1$ internally and externally respectively.
- $|z-z_1|^2+|z-z_2|^2=\alpha$, represent a circle provided $\alpha\geq\dfrac{1}{2}|z_1-z_2|^2$.
- The condition that four points z_1, z_2, z_3 and z_4 to be concyclic is $\dfrac{z_3-z_1}{z_3-z_2}\cdot\dfrac{z_4-z_2}{z_4-z_1}$ is real. Hence, the equation of a circle through three non-collinear points z_1, z_2, z_3 can be taken as $\dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real
$$\Rightarrow\ \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}=\frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$$
- $\operatorname{Arg}\left(\dfrac{z-z_1}{z-z_2}\right)=\theta$ represents
  (1) a line segment, if $\theta = 180°$
  (2) a pair of ray, if $\theta = 0°$
  (3) a part of circle, if $0° < \theta < 180°$

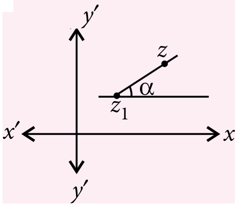
Rotation in complex number

Facts :

- Arg $z=\alpha$ represents points (other than 0) on the ray emanating from origin which makes an angle α with the positive direction of x-axis.



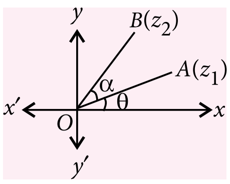
- Arg $(z-z_1)=\alpha$ represents points (other than z_1) on the ray emanating from z_1 which makes an angle α in the positive direction of x-axis (real axis).



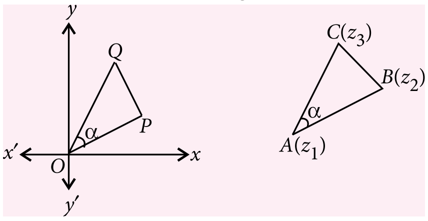
Rotation theorem

(i) If $A(z_1)$ and $B(z_2)$ are two points (complex number) such that $|z_1|=|z_2|$ then $z_2=|z_1|e^{i\alpha}$ where $\alpha=\angle AOB$, $z_1=r(\cos\theta+i\sin\theta)=re^{i\theta}$

$\therefore\ z_1\,e^{i\alpha}=r\cdot e^{i\theta}\cdot e^{i\alpha}=r\,e^{i(\theta+\alpha)}$



(ii) If $A(z_1)$, $B(z_2)$, $C(z_3)$ are affixes of points in the argand plane then in triangle ABC,



$\angle BAC=\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$

$AB=|z_2-z_1|$, $AC=|z_3-z_1|$ and $BC=|z_3-z_2|$

$\therefore\ BC=|(z_3-z_1)-(z_2-z_1)|$

Also, $\dfrac{|z_3-z_1|}{|z_2-z_1|}=\left|\dfrac{z_3-z_1}{z_2-z_1}\right|=\dfrac{AC}{AB}$

Again, $z_3-z_1=\dfrac{|z_3-z_1|}{|z_2-z_1|}(z_2-z_1)(\cos\alpha+i\sin\alpha)$

$\left(\because \overline{OQ}=\dfrac{OQ}{OP}(\overline{OP}\,e^{i\alpha})\right)$

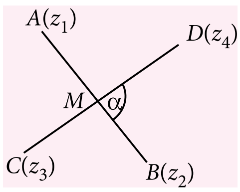
$\therefore\ \dfrac{z_3-z_1}{z_2-z_1}=\dfrac{|z_3-z_1|}{|z_2-z_1|}e^{i\alpha}$

$\Rightarrow\ \dfrac{z_3-z_1}{z_2-z_1}=\left|\dfrac{z_3-z_1}{z_2-z_1}\right|e^{i\alpha}$

- If $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$, then triangle is equilateral.

(iii) If $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$ are four complex numbers and $\angle DMB=\alpha$, then

$\dfrac{z_3-z_4}{z_1-z_2}=\left|\dfrac{z_3-z_4}{z_1-z_2}\right|e^{i\alpha}$



Logarithm of a complex number

Let $z=\alpha+i\beta$ and $|z|=\sqrt{\alpha^2+\beta^2}$

$\therefore\ \log z=\log(\alpha+i\beta)$

$=\dfrac{1}{2}\log(\alpha^2+\beta^2)+i\left(2n\pi+\tan^{-1}\dfrac{\beta}{\alpha}\right)$ (general result)

i.e. $\log z=\dfrac{1}{2}\log|z|+i\arg z$

PROBLEMS

Single Correct Answer Type

1. If $|z-5|\leq 6$ and $|\omega z+1+\omega^2|=\alpha$ (where ω is cube root of unity) then value α satisfies

(a) $0\leq\alpha\leq 10$ (b) $0\leq\alpha\leq 11$

(c) $\dfrac{\sqrt{11}}{2}-\dfrac{5}{6}\leq\alpha\leq\dfrac{\sqrt{11}}{2}+\dfrac{5}{6}$

(d) none of these

2. If $x = 3 + 5i$ (where $i^2 = -1$) and $2\left(\frac{1}{8!}+\frac{1}{2!6!}\right)+\frac{1}{4!4!}=\frac{2^a}{b!}$, then value of $x^3 - 8x^2 + 46x - 60$ is equal to

(a) $a + b$ (b) $a - b$ (c) a (d) b

3. Let X be the set of all complex numbers z such that $|z| = 1$ and define relation R on X by $z_1 R z_2$ is $\left|\operatorname{Arg}\left(\frac{z_1}{z_2}\right)\right|=\frac{5\pi}{6}$, then R is

(a) transitive (b) reflexive
(c) antisymmetric (d) none of these

4. If $f_p(\alpha)=e^{i\alpha/p^3}\cdot e^{4i\alpha/p^3}\cdot e^{9i\alpha/p^3}\cdot e^{16i\alpha/p^3}\dots\cdot e^{i\alpha/p}$ (upto p terms), where $p \in N$ and $i=\sqrt{-1}$ then $\lim_{n\to\infty} f_n(\pi)$ equals

(a) $\frac{\sqrt{3}}{2}-\frac{i}{2}$ (b) $\frac{i}{2}-\frac{\sqrt{3}}{2}$
(c) $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ (d) $-\left(\frac{i+\sqrt{3}}{2}\right)$

5. Let α and β be distinct complex numbers such that $|\alpha| = |\beta|$. If $\text{Re}(\alpha) > 0$ and $\text{Im}(\beta) < 0$, then the complex number $\frac{\alpha-\beta}{\alpha+\beta}$ may be

(a) real and negative (b) pure imaginary
(c) pure real (d) none of these

6. If z_r ($r = 1, 2, 3, ..., 2017$) are roots of the equation $\sum_{r=0}^{2017} z^r = 0$ then the value of $\left|\sum_{r=1}^{2017}\left(\frac{1}{z_r-1}\right)\right|$ equals

(a) 2018 (b) $\frac{2017}{2}$
(c) 2017 (d) none of these

7. If α be a complex number such that $|\alpha| = 1$. If the equation $\alpha z^2 + z + 1 = 0$ has a pure imaginary root then the value of $\tan(\arg \alpha)$ is

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{5}-1}{4}$
(c) $\sqrt{\frac{\sqrt{5}+1}{2}}$ (d) $\sqrt{\frac{\sqrt{5}-1}{2}}$

8. If $z+\frac{1}{z}=2\cos 15°$ then value of $z^{2017}+\frac{1}{z^{2017}}$ equals

(a) $\frac{\sqrt{3}+1}{\sqrt{2}}$ (b) $2\sin 15°$
(c) $\sin 15°$ (d) none of these

9. The complex number connected with vertices P, Q, R of ΔPQR are $e^{i\alpha}$, ω, ω^2 respectively (where ω, ω^2 are complex cube root of unity and $\cos\alpha > \text{Re}(\omega)$), then the complex number of the point where the angle bisector of P meet the circumcircle of the triangle, is

(a) $\omega\bar{\omega}$ (b) $\omega+\bar{\omega}$ (c) $e^{i\alpha}$ (d) $e^{-i\alpha}$

10. Let $A(z)$ be a variable point in the Gaussian plane such that the value $|z| = \min\{|z + 1|, |z - 1|\}$, then the sum of complex number z and its conjugate will be equal to

(a) 1 or i (b) -1 or i
(c) ω^3 or i^2 (d) none of these

More Than One Correct Answer Type

11. If $|3z + 4| = |4z - 5|$ and $|z|^2 = \alpha\text{Re}(z) + \alpha_1$, where $\alpha, \alpha_1 \in R$ and $[\cdot]$ denotes the greatest integer function then

(a) $7\alpha+\frac{7}{3}\alpha_1-29=0$ (b) $\left[\frac{\alpha}{\alpha_1}\right]=-4$
(c) $49\alpha\alpha_1 = -288$ (d) $7\alpha = 32$

12. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are complex numbers such that $|z_1| = 1$, $|z_2| = 2$ and $\text{Re}(z_1z_2) = 0$ then the pair of complex numbers $M_1=a_1+\frac{ia_2}{2}$ and $M_2 = 2b_1 + ib_2$ satisfies

(a) $|M_2| = 2$ (b) $\text{Re}(M_1M_2) = 0$
(c) $\text{Im}(M_1M_2) = 2$ (d) $|M_1| = 1$

13. If $\alpha=e^{\frac{2\pi i}{9}}$ and $\beta=e^{\frac{2\pi i}{11}}$ then

(a) $(1 + \alpha)(1 + \alpha^2) \dots (1 + \alpha^8) = 1$
(b) $(1 - \alpha^2)(1 - \alpha^4) \dots (1 - \alpha^{16}) = 9$
(c) $(i + \beta)(i + \beta^2) \dots (i + \beta^{10}) = -i$
(d) $(1 + \beta^2)(1 + \beta^4) \dots (1 + \beta^{20}) = 1$

14. Let z_1 and z_2 be two complex numbers represented by points on the circle $|z_1| = 1$ and $|z_2| = 2$ respectively, then

(a) $\max|3z_1 + z_2| = 5$ (b) $\min|z_1 - z_2| = 1$
(c) $\left|z_2+\frac{1}{z_1}\right|\le 3$ (d) $\max|3z_1 + z_2| = 4$

15. If z_1, z_2, z_3, z_4 are roots of the equation $az^4 + bz^3 + cz^2 + dz + \lambda = 0$ where $a, b, c, d, \lambda \in R$ then

(a) negative of conjugates of z_1, z_2, z_3, z_4 are also the roots of the equation.
(b) conjugate of z_1, z_2, z_3, z_4 are also the roots of the equation.
(c) z_1 is equal to at least one of the conjugate of z_1, z_2, z_3, z_4.
(d) all of the above

16. If $z_1 = 8 + 15i$ and $|z_2| = 3$ then

(a) max. of $|z_1 + iz_2| = 20$

(b) max. of $\left|\dfrac{z_1}{z_2+\dfrac{3}{z_2}}\right| = \dfrac{17}{2}$

(c) min. of $|z_1+(1+i)z_2| = 17-3\sqrt{2}$

(d) min. of $\left|\dfrac{z_1}{z_2+\dfrac{3}{z_2}}\right| = \dfrac{17}{4}$

Comprehension Type

Paragraph for Q. No. 17 to 19

Let $e^{x+iy} = \alpha + i\beta$ such that $\log(\alpha + i\beta) = x + iy$ and $x + iy = \gamma e^{i\theta}$.

On the basis of above information answer the following.

17. The value of $(-i)^{-i}$ equals

(a) $e^{(4n-1)\frac{\pi}{2}}, n \in I$ (b) $e^{i(4n-1)\frac{\pi}{2}}, n \in I$

(c) $e^{(4n+1)\frac{\pi}{2}}, n \in I$ (d) $e^{-i(4n+1)\frac{\pi}{2}}, n \in I$

18. The value of $[\sin(\log i^i)]^3 + [\cos(\log i^i)]^3$ equals

(a) 3 (b) 2

(c) 1 (d) none of these

19. If $i^{(\alpha + i\beta)} = \alpha + i\beta$ then $\alpha^2 + \beta^2$ equals

(a) $e^{(2n+1)\frac{\pi\alpha}{2}}, n \in I$ (b) $e^{-(4n+1)\pi\beta}, n \in I$

(c) $e^{(4n+1)\pi\beta}, n \in I$ (d) none of these

Matrix-Match Type

20. Match the following:

	Column-I		Column-II
A.	If $x^2 + x + 1 = 0$ then the value of $\sum_{r=1}^{100}\left(x^r + \dfrac{1}{x^r}\right)^3$ is	1.	128
B.	The value of $\dfrac{1}{3}\left[\sum_{r=1}^{10}(r-\omega)(r-\omega^2)\right]$ (where ω is the cube root of unity), is	2.	180
C.	For any integer k, let $\alpha_k = \cos\dfrac{k\pi}{7} + i\sin\dfrac{k\pi}{7}$, where $i=\sqrt{-1}$. If $m = \dfrac{\sum_{k=1}^{12}\lvert\alpha_{k+1}-\alpha_k\rvert}{\sum_{k=1}^{3}\lvert\alpha_{4k-1}-\alpha_{4k-2}\rvert}$ then the value of $45m$ equals	3.	197
D.	If $z = 1+i\sqrt{3}$ then the value of $(z)^6 + (\bar{z})^6$ equals	4.	150

21. Match the following:

	Column-I		Column-II
A.	If the complex numbers z for which $\arg\left(\dfrac{3z-6-3i}{2z-8-6i}\right) = \dfrac{\pi}{4}$ and $\lvert z-3+i\rvert = 3$ are $\left(\lambda_1 - \dfrac{\lambda_2}{\sqrt{5}}\right) + i\left(1+\dfrac{2}{\sqrt{5}}\right)$ and $\left(\lambda_1 + \dfrac{\lambda_2}{\sqrt{5}}\right) + i\left(1-\dfrac{2}{\sqrt{5}}\right)$ then the value of $\lambda_1 + \lvert\lambda_2\rvert$ equals	1.	16
B.	If $\lvert z_1\rvert = 2$, $\lvert z_2\rvert = 3$, $\lvert z_3\rvert = 4$ and $\lvert 16z_1z_2 + 9z_1z_3 + 4z_2z_3\rvert = 288$ then $\lvert z_1+z_2+z_3\rvert\lvert\bar{z}_1+\bar{z}_2+\bar{z}_3\rvert$ equals	2.	15
C.	If n be the number of common roots of the equations $x^3 + 2x^2 + 2x + 1 = 0$ and $x^{2017} + x^{2015} + 1 = 0$, then value of n^4 is	3.	$(12)^2$
D.	If $\lvert z_1\rvert = 3$, $\lvert z_2\rvert = 4$, $\lvert z_3\rvert = 5$ and $\lvert 3z_1 + 4z_2 + 5z_3\rvert = 12$ then absolute value of $\dfrac{(27z_2z_3 + 64z_1z_3 + 125z_1z_2)}{48}$ is equal to	4.	8

Integer Answer Type

22. If ω is complex cube root of unity and $a, b, c, d \in R$ such that

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=2\omega^2$$

and $\dfrac{1}{a+\omega^2}+\dfrac{1}{b+\omega^2}+\dfrac{1}{c+\omega^2}+\dfrac{1}{d+\omega^2}=2\omega$

then the value of $\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}+\dfrac{1}{1+d}=$

23. Let z_1, z_2, z_3 be three complex numbers such that they are unimodulus and $\dfrac{z_1^2}{z_2z_3}+\dfrac{z_2^2}{z_1z_3}+\dfrac{z_3^2}{z_1z_2}+1=0$ then sum of possible values of $|z_1+z_2+z_3| =$

SOLUTIONS

1. (a) : As $|\omega z + 1 + \omega^2| = \alpha$, $\therefore$ $\alpha \ge 0$ ... (i)

Again, $\alpha = |\omega z + 1 + \omega^2| = |\omega z - \omega|$

$= |\omega(z - 1)| = |z - 1|$ $(\because |\omega| = 1, 1 + \omega^2 = -\omega)$

$= |z - 5 + 4| \le |z - 5| + 4 \le 6 + 4$

$\therefore$ $\alpha \le 10$ ... (ii)

From (i) and (ii), $0 \le \alpha \le 10$

2. (d) : $2\left(\frac{1}{8!}+\frac{1}{2!6!}\right)+\frac{1}{4!4!}=\frac{2^a}{b!}$ ... (i)

Consider

$$2\left(\frac{1}{8!}+\frac{1}{2!6!}\right)+\frac{1}{4!4!}=\frac{1}{8!}\left[\frac{2!\,8!}{0!\,8!}+\frac{2!\,8!}{2!\,6!}+\frac{8!}{4!\,4!}\right]$$

$$=\frac{1}{8!}\left[2\cdot{}^8C_0+2\cdot{}^8C_2+{}^8C_4\right]$$

$$=\frac{1}{8!}\left[{}^8C_0+{}^8C_8+{}^8C_2+{}^8C_6+{}^8C_4\right]$$

$$=\frac{1}{8!}\left[{}^8C_0+{}^8C_2+{}^8C_4+{}^8C_6+{}^8C_8\right]=\frac{1}{8!}\left[2^{8-1}\right] \quad ...(ii)$$

Equating (i) and (ii), we have

$$\frac{2^7}{8!}=\frac{2^a}{b!} \Rightarrow a=7,\ b=8$$

Again, $x = 3 + 5i$

$\Rightarrow$ $x - 3 = 5i$ $\Rightarrow$ $x^2 - 6x + 34 = 0$ ... (*)

$\therefore$ $x^3 - 8x^2 + 46x - 60 = x(x^2 - 6x + 34)$
$- 2(x^2 - 6x + 34) + 8$

$= 8 = b$ (using *)

3. (d) : $\because$ $|z| = 1$ $\Rightarrow$ $z = \cos\alpha + i\sin\alpha$ $\Rightarrow$ $\text{Arg}(z) = \alpha$

$\therefore$ $\text{Arg}(z_1) = \alpha_1$ and $\text{Arg}(z_2) = \alpha_2$

Now, $z_1 R z_2 \Leftrightarrow |\text{Arg}(z_1/z_2)|=\frac{5\pi}{6}$

$z_1 R z_2 \Leftrightarrow |\text{Arg}\,z_1-\text{Arg}\,z_2|=\frac{5\pi}{6}$

$z_1 R z_2 \Leftrightarrow |\alpha_1-\alpha_2|=\frac{5\pi}{6}$

$\Leftrightarrow |\alpha_2-\alpha_1|=\frac{5\pi}{6}$

$\Leftrightarrow z_2 R z_1$

(Here $z_1 \ne z_2$, if $z_1 = z_2$, then $\frac{5\pi}{6}=0$ which is not possible)

$\Rightarrow$ R is symmetric.

4. (c) : $\because$ $f_p(\alpha)=e^{i\alpha/p^3}\cdot e^{4i\alpha/p^3}\cdot e^{9i\alpha/p^3}\ ....\ e^{p^2 i\alpha/p^3}$

$\left(\text{as } e^{i\alpha/p}=e^{ip^2\alpha/p^3}\right)$

$$=e^{\frac{i\alpha}{p^3}(1+4+9+...+p^2)}=e^{\frac{i\alpha}{p^3}\left(\frac{p(p+1)(2p+1)}{6}\right)}$$

$$=e^{\frac{i\alpha}{6}\left(1\cdot\left(1+\frac{1}{p}\right)\left(2+\frac{1}{p}\right)\right)}$$

$$\therefore \lim_{n\to\infty} f_n(\pi)=\lim_{n\to\infty} e^{\frac{i\pi}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}=e^{i\frac{\pi}{6}\times 2}=e^{\frac{i\pi}{3}}$$

$$\therefore \lim_{n\to\infty} f_n(\pi)=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

5. (b) : Let $\alpha = r(\cos\theta + i\sin\theta)$, $\forall\, \theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

and $\beta = r(\cos\phi + i\sin\phi)$, $\forall\ \phi \in (\pi, 2\pi)$

$$\therefore \frac{\alpha-\beta}{\alpha+\beta}=\frac{(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)}{(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)}$$

$$=\frac{-2\sin\frac{\theta+\phi}{2}\cdot\sin\frac{\theta-\phi}{2}+2i\cos\frac{\theta+\phi}{2}\cdot\sin\frac{\theta-\phi}{2}}{2\cos\frac{\theta+\phi}{2}\cdot\cos\frac{\theta-\phi}{2}+2i\sin\frac{\theta+\phi}{2}\cdot\cos\frac{\theta-\phi}{2}}$$

$$=\frac{2i\sin\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2}+i\sin\frac{\theta+\phi}{2}\right)}{2\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2}+i\sin\frac{\theta+\phi}{2}\right)}=i\tan\left(\frac{\theta-\phi}{2}\right)$$

= Pure imaginary

6. (b) : The given equation is

$$\sum_{r=0}^{2017} z^r=0 \Rightarrow 1+z+z^2+...+z^{2017}=0$$

$\Rightarrow$ $z^{2018} - 1 = 0,\ z \ne 1$

Let $S=\frac{1}{z_1-1}+\frac{1}{z_2-1}+\frac{1}{z_3-1}+....+\frac{1}{z_{2017}-1}$

where $z_1, z_2, ...\ z_{2017}$ are roots of the equation $z^{2018} - 1 = 0,\ z \ne 1$

Let $1 + z + z^2 + ... + z^{2017} = (z - z_1)(z - z_2) ... (z - z_{2017})$

$\Rightarrow$ $\log(1 + z + z^2 + ... + z^{2017})$

$= \log((z - z_1)(z - z_2)\ .....\ (z - z_{2017}))$ ... (*)

Differentiating with respect to z both sides of (*) we have,

$$\frac{1+2z+3z^2+...+2017\,z^{2016}}{1+z+z^2+...+z^{2017}}=\frac{1}{z-z_1}+\frac{1}{z-z_2}+...+\frac{1}{z-z_{2017}}$$

Putting $z = 1$, we have

$$\frac{1+2+3+...+2017}{1+1+...(2018\text{ times})}=-\left[\frac{1}{z_1-1}+\frac{1}{z_2-1}+...+\frac{1}{z_{2017}-1}\right]$$

$$\Rightarrow \frac{2017\times 2018}{2\times 2018}=-\sum_{r=1}^{2017}\left(\frac{1}{z_r-1}\right)$$

$$\Rightarrow \left|\sum_{r=1}^{2017}\frac{1}{z_r-1}\right|=\frac{2017}{2}$$

7. (c) : Let $z = x + iy,\ y \neq 0$

As $|\alpha| = 1 \Rightarrow \alpha = \cos\theta + i\sin\theta$, where $\theta = \arg\alpha$

Now, the given equation, $\alpha z^2 + z + 1 = 0$ reduces to

$(\cos\theta + i\sin\theta)z^2 + z + 1 = 0$

which has a pure imaginary root say $z = iy$

$\therefore\ (\cos\theta + i\sin\theta)(iy)^2 + iy + 1 = 0$

$\Rightarrow\ -(y^2\cos\theta + iy^2\sin\theta) + iy + 1 = 0$

$\Rightarrow\ y^2\cos\theta + iy^2\sin\theta = 1 + iy$

On comparing real and imaginary parts both sides, we get $y^2\cos\theta = 1$ and $y^2\sin\theta = y$

$\Rightarrow\ y^2\cos\theta = 1$ and $y\sin\theta = 1$ or $y^2\sin^2\theta = 1$

$$\Rightarrow \frac{1}{\cos\theta} = \frac{1}{\sin^2\theta} \Rightarrow \cos^2\theta + \cos\theta - 1 = 0$$

$$\Rightarrow \cos\theta = \frac{\sqrt{5}-1}{2},\ \cos\theta = -\left(\frac{\sqrt{5}+1}{2}\right) \text{ (rejected)}$$

$$\text{Now, } \tan^2\theta = \sec^2\theta - 1 = \left(\frac{\sqrt{5}+1}{2}\right)^2 - 1$$

$$\left(\because \sec\theta = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{4}\right)$$

$$= \frac{(\sqrt{5}+1)^2 - 4}{4} = \frac{\sqrt{5}+1}{2}$$

$$\Rightarrow \tan\theta = \sqrt{\frac{\sqrt{5}+1}{2}} \Rightarrow \tan(\arg\alpha) = \sqrt{\frac{\sqrt{5}+1}{2}}$$

8. (a) : $z + \dfrac{1}{z} = 2\cos 15°$

$\Rightarrow\ z^2 - 2z\cos 15° + 1 = 0 \Rightarrow z = \cos 15° \pm i\sin 15°$

$\therefore\ z = \cos 15° + i\sin 15°$ or $z = \cos 15° - i\sin 15°$

Consider $z = \cos 15° + i\sin 15°$

$$\therefore\ z^{2017} + \frac{1}{z^{2017}} = (\cos 15° + i\sin 15°)^{2017} + \frac{1}{(\cos 15° + i\sin 15°)^{2017}}$$

$= [\cos(15 \times 2017)° + i\sin(15 \times 2017)°] + [\cos(15 \times 2017)° - i\sin(15 \times 2017)°]$

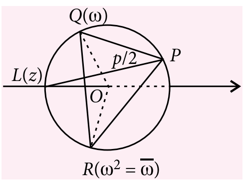
(Using $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$)

$= 2\cos 15°$ $[\because 15 \times 2017° = 30225° = (360 \times 84)° + 15°$ and $\cos(2n\pi + \theta) = \cos\theta]$

$$= \frac{\sqrt{3}+1}{\sqrt{2}} \left[\because \cos 15° = \cos(45° - 30°) = \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$

$$\therefore\ z^{2017} + \frac{1}{z^{2017}} = \frac{\sqrt{3}+1}{\sqrt{2}}$$

9. (b) : As ω and ω^2 are cube roots of unity



$\therefore\ \omega = \omega^2 = \bar{\omega}$ and

$\omega\omega^2 = \omega\bar{\omega}$

From figure,

$\angle LOQ = \angle ROL = \angle P$

$\Rightarrow\ z = \omega e^{ip}$ and $\omega^2 = \bar{\omega} = ze^{ip}$ (using rotation about O)

$\therefore\ z \cdot (ze^{ip}) = \omega e^{ip}\,\bar{\omega} \Rightarrow z^2 = \omega\bar{\omega} = \omega^3 = 1$

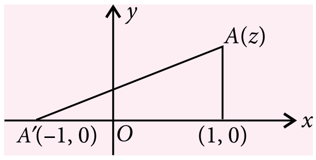
$\Rightarrow\ z = -1$ ($\because$ P and L are on opposite side of QR)

$\Rightarrow\ z = \omega + \omega^2 = \omega + \bar{\omega}$ ($\because 1 + \omega + \omega^2 = 1 + \omega + \bar{\omega} = 0$)

10. (c) : Let $z = x + iy$

$\therefore\ z - 1 = (x - 1) + iy,\ z + 1 = (x + 1) + iy$

When $|z - 1| < |z + 1|$ (or $x > 0$)



$\Rightarrow\ |z| = |z - 1|$

$\Rightarrow\ x^2 + y^2 = (x - 1)^2 + y^2$

$\Rightarrow\ (x - 1)^2 - x^2 = 0$

$\Rightarrow\ x = 1/2$

$$\text{Now, } z + \bar{z} = 2x = 2\left(\frac{1}{2}\right) = 1 = \omega^3$$

Again, when $|z - 1| > |z + 1|$ (or $x < 0$)

$\therefore\ |z| = |z + 1| \Rightarrow x^2 + y^2 = (x + 1)^2 + y^2$

$\Rightarrow\ x = -1/2$

$$\therefore\ z + \bar{z} = 2x = 2\left(-\frac{1}{2}\right) = -1 = i^2$$

Thus, $z + \bar{z} = \omega^3$ or i^2

11. (a, b, c, d) : Since, $|3z + 4| = |4z - 5|$

$\Rightarrow\ |3z + 4|^2 = |4z - 5|^2$

$\Rightarrow\ (3z + 4)(3\bar{z} + 4) = (4z - 5)(4\bar{z} - 5)$ $(\because |z|^2 = z\bar{z})$

$\Rightarrow\ 9|z|^2 + 12(z + \bar{z}) + 16 = 16|z|^2 - 20(z + \bar{z}) + 25$

$\Rightarrow\ 7|z|^2 = 32(z + \bar{z}) - 9 \Rightarrow |z|^2 = \dfrac{32}{7}\mathrm{Re}(z) - \dfrac{9}{7}$

12. (a, b, c, d) : Since $|z_1| = 1 \Rightarrow a_1^2 + b_1^2 = 1$ and

$|z_2| = 4 \Rightarrow a_2^2 + b_2^2 = 4$ ... (*)

Again, consider $z_1z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$

$\therefore\ \mathrm{Re}(z_1z_2) = 0 \Rightarrow a_1a_2 - b_1b_2 = 0$

$\Rightarrow\ a_1a_2 = b_1b_2$ ... (**)

$$\text{and } M_1M_2 = \left(a_1 + \frac{ia_2}{2}\right)(2b_1 + ib_2)$$

$$= \left(2a_1b_1 - \frac{a_2b_2}{2}\right) + i(b_1a_2 + a_1b_2) \quad \text{... (A)}$$

As $a_2^2 + b_2^2 = 4(1) = 4(a_1^2 + b_1^2)$ and $a_1a_2 = b_1b_2$

$\Rightarrow\ a_2^2 - 4a_1^2 + 4ia_1a_2 = 4b_1^2 - b_2^2 + 4ib_1b_2$

$\Rightarrow\ (a_2 + 2ia_1)^2 = (2b_1 + ib_2)^2$

$\Rightarrow\ a_2 = \pm 2b_1$ and $b_2 = \pm 2a_1$

$\Rightarrow\ a_2^2 = 4b_1^2$ and $b_2^2 = 4a_1^2$ ... (***)

$$\text{Now, } M_1 = a_1 + \frac{ia_2}{2}$$

$$\therefore\ |M_1|^2 = a_1^2 + \frac{a_2^2}{4} = a_1^2 + b_1^2 = 1$$

$\therefore\ |M_1| = \pm 1$ (using * and ***)

and $M_2 = 2b_1 + ib_2$

$\therefore\ |M_2|^2 = 4b_1^2 + b_2^2 = 4b_1^2 + 4a_1^2 = 4(a_1^2 + b_1^2) = 4$

$\Rightarrow\ |M_2| = \pm 2$

Again,

$$\text{Re}(M_1M_2)=\left(2a_1b_1-\frac{a_2b_2}{2}\right)=2a_1b_1-\frac{(2a_1)(2b_1)}{2}=0$$

and $\text{Im}(M_1M_2)=b_1a_2+a_1b_2=2(b_1^2+a_1^2)=2(1)=2$ (using *)

13. (a, b, c, d) : Here $\alpha=e^{\frac{i2\pi}{9}}$ is 9^{th} root of unity

$\therefore\ x^9-1=(x-1)(x-\alpha)(x-\alpha^2)\ ...\ (x-\alpha^8)$

$\Rightarrow\ -2=-2(-1)^8\,(1+\alpha)(1+\alpha^2)\ ...\ (1+\alpha^8)$ (by putting $x=-1$)

$\Rightarrow\ (1+\alpha)(1+\alpha^2)\ ....\ (1+\alpha^8)=1$ ... (*)

Again, $\dfrac{x^9-1}{x-1}=(x-\alpha)(x-\alpha^2)\ ....\ (x-\alpha^8)$

$\Rightarrow\ (x-\alpha)(x-\alpha^2)\ ...\ (x-\alpha^8)=1+x+x^2+\ ...\ +x^8$

$\Rightarrow\ (1-\alpha^2)(1-\alpha^2)\ ...\ (1-\alpha^8)=9$ (by putting $x=1$) .... (**)

Multiplying (*) and (**), we get

$(1-\alpha^2)(1-\alpha^4)\ ...\ (1-\alpha^{16})=1\times 9=9$

Again, $\beta=e^{\frac{i2\pi}{11}}$ is 11^{th} root of unity

$\therefore\ x^{11}-1=(x-1)(x-\beta)(x-\beta^2)\ ...\ (x-\beta^{10})$

$(-i)^{11}-1=(-i-1)(-i-\beta)(-i-\beta^2)\ ...\ (-i-\beta^{10})$ (By putting $x=-i$)

$\Rightarrow\ i-1=-(1+i)(i+\beta)(i+\beta^2)\ ...\ (i+\beta^{10})$

$\Rightarrow\ \dfrac{1-i}{1+i}=(i+\beta)(i+\beta^2)\ ....\ (i+\beta^{10})$

$\therefore\ (i+\beta)(i+\beta^2)\ ....\ (i+\beta^{10})=-i$

Taking modulus and then squaring both sides.

$1=(1+\beta^2)(1+\beta^4)\ ...\ (1+\beta^{20})$

14. (a, b, c) : As z_1 and z_2 lie on $|z|=1$ and $|z|=2$ then $|z_1|=1$ and $|z_2|=2$

$\therefore\ |3z_1+z_2|\le 3|z_1|+|z_2|=3(1)+2=5$

$\therefore\ |3z_1+z_2|\le 5$

$\therefore\ \max|3z_1+z_1|=5$

Again, $|z_1-z_2|\ge ||z_1|-|z_2||=|1-2|=1$

$\therefore\ \min|z_1-z_2|=1$

Also, $\left|z_2+\dfrac{1}{z_1}\right|\le |z_2|+\dfrac{1}{|z_1|}=2+1=3$

$\therefore\ \left|z_2+\dfrac{1}{z_1}\right|\le 3$

15. (b, c) : $az^4+bz^3+cz^2+dz+\lambda=0=0+0i$

$\therefore\ a\bar{z}^4+b\bar{z}^3+c\bar{z}^2+d\bar{z}+\lambda=0+0i$ .... (*)

(Taking conjugate both sides)

$\Rightarrow\ \bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4$ are the roots of (*)

If z_1 is pure real, then $z_1=\bar{z}_1$ and if z_1 is non-real then $\bar{z}_1$ is also root of the equation as imaginary roots always occur in conjugate pair.

16. (a, b, c, d) : We have $z_1=8+15i$, $|z_2|=3$ and

$|z_1|=\sqrt{8^2+15^2}=17$

Now, $|z_1+iz_2|\le |z_1|+|z_2|=17+3=20$

$\therefore\ |z_1+iz_2|\le 20$

$\therefore\ \max.\text{ of } |z_1+iz_2|=20$

Again, $|z_1+(1+i)z_2|\ge \big||z_1|-|1+i||z_2|\big|=|17-3\sqrt{2}|$

$\therefore\ \min.\text{ of } |z_1+(1+i)z_2|=17-3\sqrt{2}$

Again, $\left|z_2+\dfrac{3}{z_2}\right|\le |z_2|+\dfrac{3}{|z_2|}=3+\dfrac{3}{3}=4$

and $\left|z_2+\dfrac{3}{z_2}\right|\ge |z_2|-\dfrac{3}{|z_2|}=3-1=2$

$\therefore\ \max.\text{ of}\left|\dfrac{z_1}{z_2+\frac{3}{z_2}}\right|=\dfrac{|z_1|}{\left|z_2+\frac{3}{z_2}\right|}=\dfrac{17}{2}$

and $\min.\text{ of}\left|\dfrac{z_1}{z_2+\frac{3}{z_2}}\right|=\dfrac{|z_1|}{\left|z_2+\frac{3}{z_2}\right|}=\dfrac{17}{4}$

17. (a) : We know $x=e^{\log x}$ and $\log a^b=b\log a$

$\therefore\ (-i)^{(-i)}=e^{\log(-i)^{(-i)}}=e^{-i\log(-i)}$

$=e^{-i\left[\log|-i|+\left(-i\frac{\pi}{2}\right)\right]}$ ($\because \log z=\log|z|+i\arg z$)

$\Rightarrow\ (-i)^{(-i)}=e^{-i\left[\log 1+\left(-i\frac{\pi}{2}\right)+2n\pi i\right]}$, $n\in I$

$=e^{-i\left[\left(2n\pi-\frac{\pi}{2}\right)i\right]}$

$=e^{-i^2\left(2n\pi-\frac{\pi}{2}\right)}=e^{(4n-1)\frac{\pi}{2}}$

18. (d) : $[\sin(\log i^i)]^3+[\cos(\log i^i)]^3$

$=\left[\sin\log_e e^{-\pi/2}\right]^3+\left[\cos\log_e e^{-\pi/2}\right]^3$ $\left(\because (i)^i=e^{-\pi/2}\right)$

$=\left[\sin\left(-\frac{\pi}{2}\log_e e\right)\right]^3+\left[\cos\left(-\frac{\pi}{2}\log_e e\right)\right]^3$

$=\left[\sin\left(\frac{-\pi}{2}\right)\right]^3+\left[\cos\left(-\frac{\pi}{2}\right)\right]^3$ $(\because \log_e e=1)$

$=-1$

19. (b) : $i^{(\alpha+i\beta)}=\alpha+i\beta$

$\Rightarrow\ (\alpha+i\beta)\log i=\log(\alpha+i\beta)$

$\Rightarrow\ \log(\alpha+i\beta)=(\alpha+i\beta)\left(i\dfrac{\pi}{2}+2n\pi i\right)$, $n\in I$

$\Rightarrow\ \dfrac{1}{2}\log(\alpha^2+\beta^2)+i\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)=i(\alpha+i\beta)(4n+1)\dfrac{\pi}{2}$

$\Rightarrow\ \dfrac{1}{2}\log(\alpha^2+\beta^2)=\dfrac{-\beta\pi}{2}(4n+1)$ (compare real parts)

$\Rightarrow\ \alpha^2+\beta^2=e^{-(4n+1)\pi\beta}$

20. **A → 3, B → 4, C → 2, D → 1**

A. Given, $x^2 + x + 1 = 0 \Rightarrow (x - \omega)(x - \omega^2) = 0$

Now, $\sum_{r=1}^{100}\left(x^r + \frac{1}{x^r}\right)^3 = \sum_{r=1}^{100}\left(\omega^r + \frac{1}{\omega^r}\right)^3$

$$= \sum_{r=1}^{100}\left[\omega^{3r} + \frac{1}{\omega^{3r}} + 3\left(\omega^r + \frac{1}{\omega^r}\right)\right]$$

$$= \sum_{r=1}^{100}\left[2 + 3\left(\omega^r + \frac{1}{\omega^r}\right)\right] = 200 + 3\sum_{r=1}^{100}\left(\omega^r + \frac{1}{\omega^r}\right)$$

$$= 200 + 3\left[\left(\omega + \frac{1}{\omega}\right) + \left(\omega^2 + \frac{1}{\omega^2}\right) + \left(\omega^3 + \frac{1}{\omega^3}\right)\right](33 \text{ times}) + 3\left(\omega^{100} + \frac{1}{\omega^{100}}\right)$$

$$= 200 + \underbrace{3(-1 - 1 + 2)}_{33 \text{ times}} + 3(-1) = 200 - 3 = 197$$

B. $\sum_{r=1}^{10}(r - \omega)(r - \omega^2) = \sum_{r=1}^{10}(r^2 + r + 1)$

$$= \sum_{r=1}^{10} r^2 + \sum_{r=1}^{10} r + 10 = \frac{10 \times 11 \times 21}{6} + \frac{10 \times 11}{2} + 10 = 450$$

$$\therefore \frac{1}{3}\left[\sum_{r=1}^{10}(r - \omega)(r - \omega^2)\right] = 150$$

C. Since, $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = e^{\frac{k\pi}{7}i}$

$$m = \frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|} = \frac{\sum_{k=1}^{12}\left|e^{\frac{(k+1)\pi i}{7}} - e^{\frac{k\pi i}{7}}\right|}{\sum_{k=1}^{3}\left|e^{\frac{(4k-1)\pi i}{7}} - e^{\frac{(4k-2)\pi i}{7}}\right|}$$

$$= \frac{\sum_{k=1}^{12}\left|e^{\frac{k\pi i}{7}}\right|\left|e^{\frac{\pi i}{7}} - 1\right|}{\sum_{k=1}^{3}\left|e^{\frac{(4k-2)\pi i}{7}}\right|\left|e^{\frac{\pi i}{7}} - 1\right|} = \frac{\sum_{k=1}^{12} 1}{\sum_{k=1}^{3} 1} = 4$$

$$\left(\because \left|e^{\frac{(4k-2)\pi i}{7}}\right| = \left|e^{\frac{k\pi i}{7}}\right| = 1\right)$$

$\therefore$ $45m = 4 \times 45 = 180$

D. We know that

if $z = x + iy$ then $\bar{z} = x - iy$

$(x + iy)^n = C_0x^n + C_1x^{n-1}(iy) + C_2x^{n-2}(iy)^2 + \ldots + C_n(iy)^n$

and $(x - iy)^n = C_0x^n - C_1x^{n-1}(iy) + C_2x^{n-2}(iy)^2 + \ldots + C_n(-iy)^n$

Putting $n = 6$, then adding we get

$\therefore$ $(x + iy)^6 + (x - iy)^6 = 2[{}^6C_0x^6 + {}^6C_2x^4(iy)^2 + {}^6C_4x^2(iy)^4 + {}^6C_6(iy)^6]$

Putting $x = 1$ and $y = \sqrt{3}$

$$\therefore (z)^6 + (\bar{z})^6 = 2[1 - 15 \times 3 + 15 \times 9 - 27]$$
$$= 2 \times 64 = 128$$

21. **A → 4, B → 3, C → 1, D → 2**

A. Let $z = x + iy$

$\because$ $\arg\left(\frac{3z - 6 - 3i}{2z - 8 - 6i}\right) = \frac{\pi}{4}$ and $|z - 3 + i| = 3$

$\Rightarrow$ $\arg\left(\frac{3(z - 2 - i)}{2(z - 4 - 3i)}\right) = \frac{\pi}{4}$ and $(x - 3)^2 + (y + 1)^2 = 3^2$

$\Rightarrow$ $\arg\left(\frac{3}{2}\right) + \arg\left(\frac{(x - 2) + i(y - 1)}{(x - 4) + i(y - 3)}\right) = \frac{\pi}{4}$

and $x^2 + y^2 - 6x + 2y + 1 = 0$ ...(i)

$\Rightarrow$ $\arg\left[\frac{[(x - 2) + i(y - 1)][(x - 4) - i(y - 3)]}{(x - 4)^2 + (y - 3)^2}\right] = \frac{\pi}{4}$

$\Rightarrow$ $\arg\left[\left(\frac{(x - 2)(x - 4) + (y - 1)(y - 3)}{(x - 4)^2 + (y - 3)^2}\right) + i\left(\frac{(y - 1)(x - 4) - (x - 2)(y - 3)}{(x - 4)^2 + (y - 3)^2}\right)\right] = \frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\left[\frac{(y - 1)(x - 4) - (x - 2)(y - 3)}{(x - 2)(x - 4) + (y - 1)(y - 3)}\right] = \frac{\pi}{4}$

$\Rightarrow$ $\frac{(y - 1)(x - 4) - (x - 2)(y - 3)}{(x - 2)(x - 4) + (y - 1)(y - 3)} = 1$

$\Rightarrow$ $x^2 + y^2 - 8x - 2y + 13 = 0$ ...(ii)

Subtracting (ii) from (i), we get

$x + 2y - 6 = 0$ ...(iii)

Now, from (ii) and (iii), we have

$5y^2 - 10y + 1 = 0$

$\therefore$ $y = \frac{10 \pm \sqrt{80}}{10} = 1 \pm \frac{2}{\sqrt{5}}$

$\therefore$ $x = 6 - 2y = 6 - 2\left(1 \pm \frac{2}{\sqrt{5}}\right) = 4 \mp \frac{4}{\sqrt{5}} = \lambda_1 \mp \frac{\lambda_2}{\sqrt{5}}$

$\therefore$ $\lambda_1 + |\lambda_2| = 8$

B. Given $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$

and $|16z_1z_2 + 9z_1z_3 + 4z_2z_3| = 288$

$\Rightarrow$ $z_1\bar{z}_1 = 4, z_2\bar{z}_2 = 9, z_3\bar{z}_3 = 16$

Now, $|16z_1z_2 + 9z_1z_3 + 4z_2z_3| = 288$

$\Rightarrow$ $|z_1z_2z_3\bar{z}_3 + z_1z_3z_2\bar{z}_2 + z_1\bar{z}_1z_2z_3| = 288$

$\Rightarrow$ $|z_1z_2z_3||\bar{z}_3 + \bar{z}_2 + \bar{z}_1| = 288$

$\Rightarrow$ $|z_1||z_2||z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 288$

$\Rightarrow$ $2 \cdot 3 \cdot 4|\overline{z_1 + z_2 + z_3}| = 12 \times 24$

$\Rightarrow\ |z_1 + z_2 + z_3| = 12$

$\therefore\ |z_1+z_2+z_3||\bar{z}_1+\bar{z}_2+\bar{z}_3|=144$

C. $x^3 + 2x^2 + 2x + 1 = 0$

$\Rightarrow\ (x + 1)(x - \omega)(x - \omega^2) = 0$

$\Rightarrow\ x = -1, \omega, \omega^2$

Now only ω and ω^2 satisfies $x^{2017} + x^{2015} + 1 = 0$

$\therefore\ n$ = number of common roots = 2

$\therefore\ n^4 = 2^4 = 16$

D. Given $|z_1| = 3$, $|z_2| = 4$, $|z_3| = 5$ and $|3z_1 + 4z_2 + 5z_3| = 12$

$\Rightarrow\ z_1\bar{z}_1=9, z_2\bar{z}_2=16, z_3\bar{z}_3=25$

and $|\overline{3z_1+4z_2+5z_3}|=12$

Now, $|27z_2z_3 + 64z_1z_3 + 125z_1z_2|$

$=|z_1z_2z_3|\left|\frac{27}{z_1}+\frac{64}{z_2}+\frac{125}{z_3}\right|$

$=|z_1||z_2||z_3|\left|\frac{27\bar{z}_1}{z_1\bar{z}_1}+\frac{64\bar{z}_2}{z_2\bar{z}_2}+\frac{125\bar{z}_3}{z_3\bar{z}_3}\right|$

$=3\cdot4\cdot5|3\bar{z}_1+4\bar{z}_2+5\bar{z}_3|=3\cdot4\cdot5|\overline{3z_1+4z_2+5z_3}|$

$=3\cdot4\cdot5\cdot12$

Thus the modulus value of

$\frac{27z_2z_3+64z_1z_3+125z_1z_2}{48}=15$

22. (2) : As ω is complex cube root of unity

$\therefore\ 1 + \omega + \omega^2 = 0$

We have, $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=2\omega^2$

and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=2\omega$

$\Rightarrow\ \frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=\frac{2}{\omega}$ ...(i)

and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=\frac{2}{\omega^2}$ ...(ii)

$\Rightarrow\ \omega, \omega^2$ are roots of the equation

$\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}+\frac{1}{d+x}=\frac{2}{x}$ ...(*)

$\Rightarrow\ x\,\Sigma(b + x)(c + x)(d + x) = 2(a + x)(b + x)(c + x)(d + x)$

$\Rightarrow\ 2x^4 + 0x^2 + (a + b + c + d)x^3 - (abc + abd + acd + bcd)x - 2abcd = 0$ ...(**)

which is fourth degree equation in x, so it has four roots. Two of its roots are ω and ω^2, let other two roots be α, β *i,e.*, its four roots are $\alpha, \beta, \omega, \omega^2$.

Now, coefficient of x^2 in (**) = 0

i.e., Sum of roots taken two at a time is zero.

$\Rightarrow\ \omega\omega^2 + \alpha\omega + \alpha\omega^2 + \beta\omega + \beta\omega^2 + \alpha\beta = 0$

$\Rightarrow\ \omega\omega^2 + (\alpha + \beta)(\omega + \omega^2) + \alpha\beta = 0$

$\Rightarrow\ 1 + (\alpha + \beta)(-1) + \alpha\beta = 0$

$\Rightarrow\ (\alpha - 1)(\beta - 1) = 0 \Rightarrow \alpha = \beta = 1$

$\therefore$ From (i) or (ii) or (*), we have

$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=2$

23. (3) : As z_1, z_2, z_3 are unimodulus

$\therefore\ |z_1| = |z_2| = |z_3| = 1$

$\therefore |z_1|^2=1 \Rightarrow z_1\bar{z}_1=1 \Rightarrow z_1=\frac{1}{\bar{z}_1}$

and we get similar results for the other two

Again, $\frac{z_1^2}{z_2z_3}+\frac{z_2^2}{z_1z_3}+\frac{z_3^2}{z_1z_2}+1=0$

$\Rightarrow\ z_1^3+z_2^3+z_3^3+z_1z_2z_3=0$

$\Rightarrow\ (z_1^3+z_2^3+z_3^3-3z_1z_2z_3)=-4z_1z_2z_3$

$\Rightarrow\ (\Sigma z_1)[(\Sigma z_1)^2 - \Sigma z_1z_2] = -4z_1z_2z_3$ ...(*)

$(\because a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca))$

Let $\Sigma z_1 = z \Rightarrow (\Sigma z_1)^2 = z^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

and $\Sigma\bar{z}_1=\bar{z}$

$\therefore$ From (*), we get, $z(z^2 - 3\Sigma z_1z_2) = -4z_1z_2z_3$

$\Rightarrow\ z^3 = 3z\Sigma z_1z_2 - 4z_1z_2z_3$

$=z_1z_2z_3\left[\frac{3z\Sigma z_1z_2-4z_1z_2z_3}{z_1z_2z_3}\right]$

$=z_1z_2z_3\left[3z\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)-4\right]$

$=z_1z_2z_3[3z(\bar{z}_1+\bar{z}_2+\bar{z}_3)-4]\ \left(\because z\bar{z}=1 \Rightarrow z=\frac{1}{\bar{z}}\right)$

$=z_1z_2z_3[3(z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3)-4]$

$\Rightarrow\ z^3 = z_1z_2z_3[3|z|^2 - 4]$ ....(A)

If $|z|\geq\frac{2}{\sqrt{3}}$, we have

$\therefore\ |z|^3 = 3|z|^2 - 4$ $(\because |z_1| = |z_2| = |z_3| = 1)$

$\Rightarrow\ |z|^3 - 3|z|^2 + 4 = 0$

$\Rightarrow\ (|z| - 2)(|z|^2 - |z| - 2) = 0$

$\Rightarrow\ (|z| - 2)^2(|z| + 1) = 0$

$\Rightarrow\ |z| = 2$ or $|z| = -1$ (rejected)

Now, again from equation (A) if $|z|<\frac{2}{\sqrt{3}}$ we have

$|z|^3 = 4 - 3|z|^2 \Rightarrow |z|^3 + 3|z|^2 - 4 = 0$

$\Rightarrow\ (|z| - 1)(|z| + 2)^2 = 0$

$\Rightarrow\ |z| = 1$ (as $|z| \neq -2$ is impossible)

Thus $|z| = \{1, 2\}$ where $z = z_1 + z_2 + z_3$

$\therefore\ |z_1 + z_2 + z_3| = 1 + 2 = 3$

BRAIN @ WORK

Topics Covered : Complex Numbers and Quadratic Equations, Permutations and Combinations, Binomial Theorem

1. The value of

$$\frac{C_0}{1\cdot 3}-\frac{C_1}{2\cdot 3}+\frac{C_2}{3\cdot 3}-\frac{C_3}{4\cdot 3}+....+(-1)^n\frac{C_n}{(n+1)\cdot 3} \text{ is}$$

(a) $\frac{3}{n+1}$ (b) $\frac{n+1}{3}$

(c) $\frac{1}{3(n+1)}$ (d) None of these

2. The largest term in the expansion of $(3 + 2x)^{50}$, where $x = 1/5$, is

(a) 5^{th} (b) 6^{th} (c) 8^{th} (d) 9^{th}

3. Sum of coefficients of terms which contains integral powers of x in the expansion $(2+\sqrt{x})^{30}$ is

(a) $\frac{2^{30}+1}{2}$ (b) $\frac{2^{30}-1}{2}$

(c) $\frac{3^{30}-1}{2}$ (d) $\frac{3^{30}+1}{2}$

4. Number of terms in the expansion of $(1 + x)^{101}(1 + x^2 - x)^{100}$ is

(a) 302 (b) 301 (c) 202 (d) 101

5. If $x=(5\sqrt{5}+11)^{2n+1}$, then the value of $[x]$ is

(a) even (b) odd

(c) even or odd depending upon the value of n

(d) cannot be determined

6. If $(1 + x)(1+ x + x^2)(1 + x + x^2 + x^3)....(1 + x + x^2 + x^3 +...+ x^n) = a_0 + a_1x + a_2x^2 + ...+ a_mx^m$, then

$$\sum_{r=0}^{m} a_r =$$

(a) 1 (b) n (c) $(n + 1)!$ (d) $n!$

7. In the binomial expansion $(2^{1/3} + 3^{-1/3})^n$, if the ratio of the seventh term from the beginning of the expansion to the seventh term from its end is 1/6, then n =

(a) 6 (b) 9 (c) 12 (d) 15

8. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \dots + a_{10}x^{10}$, then $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is equal to

(a) 3^{10} (b) 2^{10}

(c) 2^9 (d) None of these

9. If the number of terms in $\left(x+1+\frac{1}{x}\right)^n$ $(n \in I^+)$ is 401, then n is greater than

(a) 201 (b) 200 (c) 199 (d) 202

10. The last two digits of the number $(23)^{14}$ are

(a) 01 (b) 03

(c) 09 (d) none of these

11. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ be such that min $f(x) >$ max $g(x)$, then the relation between b and c is

(a) non real values of b and c

(b) $0 < c < \sqrt{2}b$

(c) $|c| < \sqrt{2}|b|$

(d) $|c| > \sqrt{2}|b|$

12. Number of solutions of equation $|x - 1| = e^x$ is

(a) 2 (b) 3 (c) 0 (d) 1

13. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients and $f(2i) = f(2 + i) = 0$. The value of $(a + b + c + d)$ equals to

(a) 1 (b) 4 (c) 9 (d) 10

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14. Number of values of x for which $\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$ is
(a) 2 (b) 3
(c) 1 (d) no value of x

15. Let a, b, c be the three sides of a triangle, then the quadratic equation $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$ has
(a) both roots positive
(b) both roots negative
(c) both roots imaginary
(d) None of these

16. If $9^x + a\cdot 3^x + 1 = 0$ has no real solution for x then set of all real values of a is
(a) $(-2, 2)$ (b) $(-2, \infty)$
(c) $(-\infty, 2)$ (d) None of these

17. If $6^x + 6^{x+1} = 2^x + 2^{x+1} + 2^{x+2}$, then number of solutions of x is
(a) one (b) two
(c) infinite (d) None of these

18. If the sides of a triangle are the roots of $x^3 - 2x^2 - x - 16 = 0$ then product of the inradius and circumradius is
(a) 3 (b) 4 (c) 5 (d) 6

19. Number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}$ is
(a) 0 (b) 1 (c) 2 (d) 3

20. The solution set of the inequation
$\sqrt{2 + x - x^2} > x - 4$ is
(a) $-1 \le x \le 2$ (b) $(-\infty, \infty)$
(c) $(-1, 4)$ (d) $[2, \infty)$

21. The equation $\cos^8 x + b\cos^4 x + 1 = 0$ will have a solution if b belongs to
(a) $(-\infty, 2]$ (b) $[2, \infty)$
(c) $(-\infty, -2]$ (d) None of these

22. The sum of the factors of 8! which are odd and are of the form $(3m + 2)$ where $m \in N$ is
(a) 8 (b) 45
(c) 35 (d) None of these

23. Four couples (husband and wife) decide to form a committee of four members. The number of different committees that can be formed in which no couple finds a place is
(a) 10 (b) 12 (c) 14 (d) 16

24. The total numbers of integral solutions for (x, y, z) such that $xyz = 24$ is
(a) 36 (b) 90 (c) 96 (d) 120

25. The number of five digit telephone number having atleast one of their digits repeated is
(a) 90000 (b) 100000
(c) 30240 (d) 69760

26. Four normal dice are rolled once. The number of possible outcomes in which at least one die shows up 2 is
(a) 216 (b) 648
(c) 625 (d) 671

27. A closet has 5 pairs of shoes. The number of ways in which 4 shoes can be chosen from it so that there will be no complete pair is
(a) 80 (b) 160
(c) 200 (d) None of these

28. A seven-digit number made up of all distinct digits 8, 7, 6, 4, 2, x and y is divisible by 3. Then possible number of ordered pair (x, y) is
(a) 4 (b) 8
(c) 2 (d) None of these

29. The maximum number of points of intersection of 7 straight lines and 5 circles when 3 straight lines are parallel and 2 circles are concentric, is/are
(a) 106 (b) 96
(c) 90 (d) None of these

30. A five letter word is to be formed such that the letters appearing in the odd numbered positions are taken from the letters which appear without repetition in the word 'MATHEMATICS'. Further the letters appearing in the even numbered positions are taken from the letters which appear with repetitions in the same word MATHEMATICS. In how many different ways the five letter word can be formed ?
(a) 390 (b) 600 (c) 540 (d) 450

31. The number of ways to give 16 different things to three persons A, B, C so that B gets 1 more than A and C gets 2 more than B, is
(a) $\frac{16!}{4!5!7!}$ (b) $4!\,5!\,7!$
(c) $\frac{16!}{3!5!8!}$ (d) $3!\,5!\,8!$

32. The value of $\sum_{n=0}^{100} i^{n!}$ equals (where $i=\sqrt{-1}$)

(a) -1 (b) i

(c) $2i + 95$ (d) $97 + i$

33. If z is a complex number satisfying $|z^2 - 1| = |z|^2 + 1$ then z lies on a

(a) circle (b) parabola

(c) ellipse (d) None of these

34. If z is a complex number satisfying $|z| = 1$, the points $1 + 2z$ lie on a

(a) circle with radius 1 and centre (0, 1)

(b) circle with radius 2 and centre (1, 0)

(c) straight line

(d) circle with radius 3 and centre at the origin

35. $Z \in C$ satisfies the condition $|Z| \geq 3$. Then the least value of $\left|Z+\frac{1}{Z}\right|$ is

(a) $\frac{3}{8}$ (b) $\frac{8}{5}$ (c) $\frac{8}{3}$ (d) $\frac{5}{8}$

36. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are roots of the equation $z^5+z^4+z^3+z^2+z+1=0$ then $\prod_{i=1}^{5}(2-\alpha_i)$ is equal to

(a) 63 (b) 31 (c) 32 (d) 64

37. If $|z| = 2$ and $\frac{z_1-z_3}{z_2-z_3}=\frac{z-2}{z+2}$ then z_1, z_2, z_3 will be vertices of a/an

(a) equilateral triangle

(b) acute angled triangle

(c) right angled triangle

(d) square

38. If $|z-4+3i| \leq 1$ and α and β be the least and greatest values of $|z|$ and k be the least value of $\frac{x^4+x^2+4}{x}$ on the interval $(0, \infty)$ then k is equal to

(a) α (b) β

(c) $\alpha + \beta$ (d) None of these

SOLUTIONS

1. **(c):** $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$

On integrating from -1 to 0, we get

$$C_0-\frac{C_1}{2}+\frac{C_2}{3}\ldots+(-1)^n\frac{C_n}{n+1}=\frac{1}{n+1}$$

Hence required $=\frac{1}{3(n+1)}$

2. **(b) :** Greatest term in the expansion of $(x + y)^n$ is k^{th} term where $k=\left[\frac{(n+1)y}{x+y}\right]$

In the present case

$$k=\left[\frac{(50+1)(2x)}{3+2x}\right]=\left[\frac{(51)(2/5)}{3+2/5}\right]=\left[\frac{102}{17}\right]=6$$

Thus, 6^{th} term is the largest term.

3. **(d) :**

$$(2+\sqrt{x})^{30} = {}^{30}C_0 2^{30} + {}^{30}C_1 2^{29}\sqrt{x} + {}^{30}C_2 2^{28}(\sqrt{x})^2 + \ldots + {}^{30}C_{30}(\sqrt{x})^{30}$$

$\Rightarrow$ $S = {}^{30}C_0 2^{30} + {}^{30}C_2 2^{28} + {}^{30}C_4 2^{26} + \ldots.. + {}^{30}C_{30}$

$(x + 1)^{30} = {}^{30}C_0\, x^{30} + {}^{30}C_1\, x^{29} + \ldots. + {}^{30}C_{30}$ ...(i)

Put $x = 2$ in (i), we get

$3^{30} = {}^{30}C_0 2^{30} + {}^{30}C_1 2^{29} + \ldots\ldots. + {}^{30}C_{30}$ ...(ii)

Put $x = -2$ in (i), we get

$1 = {}^{30}C_0 2^{30} - {}^{30}C_1 2^{29} + \ldots\ldots + {}^{30}C_{30}$ ...(iii)

Adding (ii) & (iii), we get

$$\frac{3^{30}+1}{2} = {}^{30}C_0 2^{30} + {}^{30}C_2 2^{28} + \ldots. + {}^{30}C_{30} = S$$

4. **(c) :** $(1 + x)\ \{(1 + x)\ (1 + x^2 - x)\}^{100}$

$= (1 + x)\ (1 + x^3)^{100}$

Total number of terms = 101 + 101 = 202 terms

5. **(a) :** Let $(5\sqrt{5}+11)^{2n+1}=I+f$

Now suppose $F=(5\sqrt{5}-11)^{2n+1}$

$I+f-F=(5\sqrt{5}+11)^{2n+1}-(5\sqrt{5}-11)^{2n+1}$

$= 2[{}^{2n+1}C_1\ (125)^n + {}^{2n+1}C_3\ (125)^{n-1}\ 11^3 + \ldots]$

$\Rightarrow$ $I + f - F = 2k$

$\Rightarrow$ $f - F = 2k - I$ is an integer

Now, $0 \leq f < 1,\ \ 0 < F < 1, -1 < -F < 0$

$-1 < f - F < 1 \Rightarrow f - F = 0$

$\therefore$ $I = 2k \Rightarrow$ even integer

6. **(c) :** $(1 + x)\ (1 + x + x^2)\ (1+ x + x^2 + x^3) \ldots.$

$(1 + x + x^2 + \ldots. + x^n) = a_0 + a_1x + a_2x^2 + \ldots..+a_mx^m$

Substituting $x = 1$, we get

$$1\times(2)(3)(4)\ldots.(n+1)=\sum_{r=0}^{m}a_r=(n+1)!$$

7. **(b) :** $T_{r+1} = {}^nC_r\, a^{n-r} \cdot b^r$ where $a = 2^{1/3}$ and $b = 3^{-1/3}$

T_7 from beginning $= {}^nC_6\ a^{n-6}\ b^6$

and T_7 from end $= {}^nC_6\ b^{n-6}\ a^6$

$$\Rightarrow \frac{a^{n-12}}{b^{n-12}}=\frac{1}{6} \Rightarrow 2^{\frac{n-12}{3}}\cdot 3^{\frac{n-12}{3}}=6^{-1}$$

$\Rightarrow$ $n - 12 = -3 \Rightarrow n = 9$

8. (b) : Put $x = i$ and $-i$, we get

$(1 + i)^{10} = a_0 + a_1 i - a_2 - a_3 i + a_4 + \dots -a_{10}$ ...(i)

$(1 - i)^{10} = a_0 - a_1 i - a_2 + a_3 i + a_4 + \dots -a_{10}$ ...(ii)

Multiplying (i) and (ii), we get

$2^{10} = (a_0 - a_2 + a_4 -...- a_{10})^2 + (a_1 - a_3 + a_5 \dots a_9)^2$

9. (b) : We have, $\left(x+1+\frac{1}{x}\right)^n = \frac{(1+x+x^2)^n}{x^n}$

$(1 + x + x^2)^n$ is the form $a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$ which contains $2n + 1$ terms

$2n + 1 = 401 \Rightarrow n = 200$

10. (c) : $(23)^{14} = (529)^7 = (530 - 1)^7$

$= {}^7C_0\,(530)^7 - {}^7C_1(530)^6 + \dots - {}^7C_5(530)^2 + {}^7C_6(530) - 1$

$= {}^7C_0(530)^7 - {}^7C_1(530)^6 + \dots + 3710 - 1$

$= 100m + 3709$

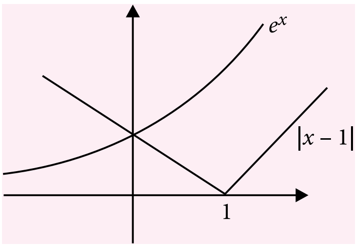
$\therefore$ Last two digits are 09

11. (d) : $\min f(x) > \max g(x)$

$\Rightarrow\ 2c^2 - b^2 > b^2 + c^2 \Rightarrow c^2 - 2b^2 > 0$

$\Rightarrow\ |c| > \sqrt{2}\,|b|$

12. (d) : From the graph it is obvious that the system has one solution.



13. (c) : If a polynomial has real coefficients, then roots occur in complex conjugate and roots are $2i$, $-2i$, $2 + i$, $2 - i$.

Hence, $f(x) = (x + 2i)(x - 2i)(x - 2 - i)(x - 2 + i)$

$f(1) = (1 + 2i)(1 - 2i)(1 - 2 - i)(1 - 2 + i)$

$\Rightarrow\ f(1) = 5 \times 2 = 10$

Also, $f(1) = 1 + a + b + c + d$

$\therefore\ 1 + a + b + c + d = 10 \Rightarrow a + b + c + d = 9.$

14. (a) : We have, $\frac{8^x+27^x}{12^x+18^x} = \frac{7}{6}$

$$\Rightarrow \frac{(8)^x}{(12)^x}\frac{(1+(27/8)^x)}{(1+(18/12)^x)} = \frac{7}{6}$$

$$\Rightarrow \left(\frac{2}{3}\right)^x\left(\frac{1+(3/2)^{3x}}{1+(3/2)^x}\right) = \frac{7}{6}$$

Let $\left(\frac{3}{2}\right)^x = t$

$$\therefore \frac{1+t^3}{t(1+t)} = \frac{7}{6} \quad \because t+1 \neq 0$$

$$\Rightarrow \frac{(1+t)(t^2+1-t)}{t(1+t)} = \frac{7}{6}$$

$$\Rightarrow \frac{t^2+1-t}{t} = \frac{7}{6} \Rightarrow t = \frac{2}{3} \text{ or } \frac{3}{2}$$

15. (c) : $\because b^2 + c^2 - a^2 = 2bc\cos A$

$\therefore$ The given equation becomes,

$b^2x^2 + 2bc\cos A\ x + c^2 = 0$

Now, $D = (2bc\cos A)^2 - 4b^2c^2 = 4b^2c^2(\cos^2 A - 1) < 0$

16. (b) : $\because 3^x + 3^{-x} = -a$

For no solution $-a < 2$

$\Rightarrow\ a > -2$

17. (a) : If $6^x(1 + 6) = 2^x(1 + 2 + 4) \Rightarrow 7\cdot6^x = 7\cdot2^x$

$\Rightarrow\ 6^x = 2^x \Rightarrow x = 0$ is only solution.

18. (b) : Let the sides of the triangle be a, b, c. Since a, b, c are roots of $x^3 - 2x^2 + x - 16 = 0$

$\therefore$ Sum of the roots $a + b + c = 2 \Rightarrow 2s = 2$ *i.e.* $s = 1$ where s is the semi-perimeter of the triangle.

Product of the roots *i.e.* $a \cdot b \cdot c = 16$

$$\therefore R\cdot r = \frac{abc}{4\Delta}\cdot\frac{\Delta}{s} = \frac{abc}{4s} = 4$$

19. (d) : We have, $\frac{x+2}{x^2+1} - \frac{1}{2} > 0$

$$\Rightarrow \frac{-x^2-1+2x+4}{2(x^2+1)} > 0 \Rightarrow \frac{3+2x-x^2}{2(x^2+1)} > 0$$

Since denominator is positive

$\therefore\ 3 + 2x - x^2 > 0 \Rightarrow -1 < x < 3 \Rightarrow x = 0, 1, 2$

20. (a) : Since $\sqrt{2+x-x^2}$ is defined only in $-1 \le x \le 2$ and it is meaningless for other values of x.

R.H.S. $= x - 4 < 0$ in $-1 \le x \le 2$

21. (c) : Given equation can be written as

$$b = -\left(\cos^4 x + \frac{1}{\cos^4 x}\right)$$

$\le -2\ \forall\ x \in R$

$\Rightarrow\ b \in (-\infty, -2]$

22. (d) : $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. For odd factors of the form $3m + 2$ we take combination of 5 and 7.

23. (d) : The number of ways to form committee of four in which no couple finds their place is

$4 \times 2 \times 2 \times 1 = 16$

24. (d)

25. (d) : Total telephone nos. can be formed = 10^5
Nos. in which no repetition = ${}^{10}P_5$ = 30240
So in which repetition are there $=10^5 - 30240 = 69760$

26. (d) : Possible outcomes = $6^4 - 5^4 = 671$

27. (a) : Select 4 pairs in ${}^5C_4$ = 5 ways. Now select exactly one shoe from each of the pair selected in $({}^2C_1)^4$ ways.
This will fulfill the condition.
$\Rightarrow$ Required answer = 5 × 16 = 80

28. (b) : 8, 7, 6, 4, 2, x and y
Any number is divisible by 3 if sum of digits is divisible by 3
i.e. $x + y + 27$ is divisible by 3.
x and y can take values from 0, 1, 3, 5, 9
Possible pairs (5, 1), (3, 0), (9, 0), (9, 3), (1, 5), (0, 3), (0, 9) and (3, 9)

29. (a) : Points of intersection due to 7 straight lines $= {}^7C_2 - 3 = 18$
Two concentric circles can intersect these 7 lines at maximum = 14 + 14 = 28 points
Third circle can intersect the given system at maximum
= 14 + 2 + 2 = 18 points
Fourth circle can intersect the system at maximum
= 14 + 2 + 2 + 2 = 20 points
For fifth circle = 14 + 2 + 2 + 2 + 2 = 22 points
Maximum no. of points of intersection
= 18 + 28 + 18 + 20 + 22 = 106

30. (c) : In word MATHEMATICS
H, E, C, I and S – without repetition
M, A, T – occurs twice
5 letters can be placed on 3 places in ${}^5P_3$ ways.
Again even places 2nd and 4th position can be filled by the three letter M, A and T.
Even places can be filled in two ways.
(1) Choose 1 letter from 3 given letters M, A and T ${}^3C_1$ ways
(2) Choose 2 letters from 3 given letters M, A and T and arrange them in 2! ways ${}^3C_1 \times 2!$ ways
Total ways ${}^3C_1 + {}^3C_1 \times 2! = 9$ ways
Required number of ways = ${}^5P_3 \times 9 = 540$ ways.

31. (a) : No. of things given to A is 4
No. of things given to B is 5
No. of things given to C is 7

$$\therefore \text{ No. of ways} = \frac{16!}{4!5!7!}$$

32. (c) : $S = \sum_{n=0}^{100} (i)^{\underline{|n}}$

$$S = (i)^{\underline{|0}} + (i)^{\underline{|1}} + (i)^{\underline{|2}} + \ldots$$

$$= i + i - 1 + i^6 + i^{24} + (i)^{\underline{|5}} + (i)^{\underline{|6}} + \ldots + (i)^{\underline{|100}} = 95 + 2i$$

33. (d) : On putting $z = x + iy$ the equation is same as $|x^2 - y^2 + 2ixy - 1| = x^2 + y^2 + 1$
$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2 \Rightarrow x = 0$
$\Rightarrow$ z lies on imaginary axis, so (a), (b), (c) are ruled out.

34. (b) : Let $w = 1 + 2z$. Then $w - 1 = 2z$
$\therefore$ $|w - 1| = 2|z| = 2$ for points on $|z| = 1$
$\therefore$ The locus of w is a circle with centre at (1, 0) and radius 2 when $|z| = 1$

35. (c) : $\left|Z + \frac{1}{Z}\right| = \left|Z - \left(-\frac{1}{Z}\right)\right| \ge |Z| - \left|-\frac{1}{Z}\right| \ge 3 - \frac{1}{3} = \frac{8}{3}$

36. (a) : $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$
$\Rightarrow (z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4)(z - \alpha_5)$
$= z^5 + z^4 + z^3 + z^2 + z + 1$
Putting $z = 2$
$\Rightarrow (2 - \alpha_1)(2 - \alpha_2)(2 - \alpha_3)(2 - \alpha_4)(2 - \alpha_5)$
$= 32 + 16 + 8 + 4 + 2 + 1$

$$\therefore \prod_{i=1}^{5}(2 - \alpha_i) = 63$$

37. (c) : $\arg\left(\frac{z-2}{z+2}\right) = \pm\frac{\pi}{2} \Rightarrow \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \pm\frac{\pi}{2}$
$\Rightarrow z_1, z_2, z_3$ are vertices of a right angled triangle.

38. (b) : From $|z - 4 + 3i| \le 1 \Rightarrow |z|_{max} = 6$, $|z|_{min} = 4$
So $\alpha = 4$, $\beta = 6$

$$y = \frac{x^4 + x^2 + 4}{x} = x^3 + x + \frac{4}{x}$$

For y to be least $x = 1 \Rightarrow y = 6 \Rightarrow y = \beta \Rightarrow k = \beta$

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MPP-4 CLASS XII ANSWER KEY

1.	(d)	**2.**	(d)	**3.**	(b)	**4.**	(d)	**5.**	(b)
6.	(a)	**7.**	(a,b,d)	**8.**	(a, b)	**9.**	(b, d)	**10.**	(a,c)
11.	(b,c,d)	**12.**	(c, d)	**13.**	(b, c)	**14.**	(a)	**15.**	(c)
16.	(a)	**17.**	(5)	**18.**	(3)	**19.**	(9)	**20.**	(3)

CONCEPT BOOSTERS

Class XI

Quadratic Equations and Inequations

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

POLYNOMIAL

Algebraic expression containing many terms of the form cx^n, n being a non-negative integer is called a polynomial, *i.e.*, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots a_{n-1}x^{n-1} + a_nx^n$, where x is a variable, $a_1, a_2, a_3, \ldots, a_n$ are constants and $a_n \neq 0$.

The expression $4x^4 + 3x^3 - 7x^2 + 5x + 3$, $3x^3 + x^2 - 3x + 5$ are examples of polynomial.

- **Real polynomial**
  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ is called real polynomial of real variable x with real coefficients.
  $3x^3 - 4x^2 + 5x - 4$, $x^2 - 2x + 1$ etc. are real polynomials.
- **Complex polynomial**
  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ is called complex polynomial of complex variable x with complex coefficients.
  $3x^2 - (2 + 4i)x + (5i - 4)$, $x^3 - 5ix^2 + (1 + 2i)x + 4$ etc. are complex polynomials.
- **Degree of a polynomial :** Highest power of variable x in a polynomial is called degree of polynomial.
  $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ is a n degree polynomial.
  $f(x) = 4x^3 + 3x^2 - 7x + 5$ is a 3 degree polynomial.

A polynomial of second degree is generally called a quadratic polynomial. Polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials respectively.

- **Polynomial equation :** If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.
- **Roots of a quadratic equation :** The values of variable x which satisfy the quadratic equation is called roots of quadratic equation.

Factorization method

Let $ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$

Then, $x = \alpha$ and $x = \beta$ will satisfy the given equation. Hence, factorize the equation and equating each factor to zero gives roots of the equation.

For example, $3x^2 - 2x - 1 = 0 \Rightarrow (x - 1)(3x + 1) = 0$
$\Rightarrow x = 1, -1/3$ are zeroes of the given equation.

By completing the perfect square

Consider the quadratic equation,

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$, we get

$$\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right] = 0$$

which gives, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence, the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a},\ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

Note : Every quadratic equation has two and only two roots.

NATURE OF ROOTS

In a quadratic equation $ax^2 + bx + c = 0$, let us suppose that a, b, c are real and $a \neq 0$. The following is true about the nature of its roots.

- The equation has real and distinct roots if and only if Discriminant $(D) \equiv b^2 - 4ac > 0$.
- The equation has real and coincident (equal) roots if and only if $D \equiv b^2 - 4ac = 0$.
- The equation has complex roots of the form $\alpha \pm i\beta, \alpha \neq 0, \beta \neq 0 \in R$ if and only if $D \equiv b^2 - 4ac < 0$.
- The equation has rational roots if and only if $a, b, c \in Q$ (the set of rational numbers) and $D \equiv b^2 - 4ac$ is a perfect square (of a rational number).
- The equation has (unequal) irrational (surd form) roots if and only if $D \equiv b^2 - 4ac > 0$ and not a perfect square even if a, b and c are rational. In this case if $p+\sqrt{q}$, (p and q are rational) is an irrational root, then $p-\sqrt{q}$ is also a root.
- $\alpha + i\beta$ ($\beta \neq 0$ and $\alpha, \beta \in R$) is a root if and only if its conjugate $\alpha - i\beta$ is a root, that is complex roots occur in pairs in a quadratic equation. In case the equation is satisfied by more than two complex numbers, then it reduces to an identity. $0 \cdot x^2 + 0 \cdot x + 0 = 0$, *i.e.*, $a = b = c = 0$.

HIGHER DEGREE EQUATIONS

The equation,

$p(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ ... (i)

where the coefficients $a_0, a_1, ..., a_n \in R$(or C) and $a_0 \neq 0$ is called an equation of n^{th} degree, which has exactly n roots $\alpha_1, \alpha_2, ..., \alpha_n \in R$(or C) then we can write

$$p(x) = a_0(x - \alpha_1)(x - \alpha_2) ... (x - \alpha_n)$$

$$= a_0\{x^n - (\Sigma\alpha_1)x^{n-1} + (\Sigma\alpha_1\alpha_2)x^{n-2} - ... + (-1)^n\alpha_1\alpha_2....\alpha_n\} \quad ... \text{(ii)}$$

Comparing (i) and (ii), we get

$$\Sigma\alpha_1 = \alpha_1 + \alpha_2 + ... + \alpha_n = -\frac{a_1}{a_0}$$

$$\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + ... + \alpha_{n-1}\alpha_n = \frac{a_2}{a_0}$$

and so on and $\alpha_1\alpha_2.....\alpha_n = (-1)^n \frac{a_n}{a_0}$

RELATION BETWEEN ROOTS AND COEFFICIENTS

- **Quadratic equation :** If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ then

  Sum of roots, $S = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coeff. of } x}{\text{Coeff. of } x^2}$

  Product of roots, $P = \alpha \cdot \beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coeff. of } x^2}$
- **Cubic equation :** If α, β, γ are the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$, then

  $\alpha+\beta+\gamma = -\frac{b}{a}$; $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$; $\alpha\beta\gamma = -\frac{d}{a}$
- **Biquadratic equation :** If $\alpha, \beta, \gamma, \delta$ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, $(a \neq 0)$, then

  $\alpha+\beta+\gamma+\delta = -\frac{b}{a}$

  $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$

  $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}$

  $\alpha\beta\gamma\delta = \frac{e}{a}$
- **Formation of a polynomial equation from given roots :** If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the roots of a polynomial equation of degree n, then the equation is

  $x^n - \sigma_1x^{n-1} + \sigma_2x^{n-2} - \sigma_3x^{n-3} + ... + (-1)^n\sigma_n = 0$

  where $\sigma_r = \sum \alpha_1\alpha_2.....\alpha_r$.
- **Quadratic equation :** A quadratic equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$.

  $\therefore \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$

  i.e. x^2 – (sum of roots)x + (products of roots) = 0

  $\therefore \quad x^2 - Sx + P = 0$
- **Cubic equation :** If α, β, γ are the roots of a cubic equation, then the equation is

  $x^3 - \sigma_1x^2 + \sigma_2x - \sigma_3 = 0$ or

  $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$
- **Biquadratic equation :** If $\alpha, \beta, \gamma, \delta$ are the roots of a biquadratic equation, then the equation is

  $x^4 - \sigma_1x^3 + \sigma_2x^2 - \sigma_3x + \sigma_4 = 0$

  or $x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta = 0$
- **Symmetric function of the roots :** A function of α and β is said to be a symmetric function, if it remains unchanged when α and β are interchanged.

  For example, $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of α and β whereas $\alpha^2 - \beta^2 + 3\alpha\beta$ is not a symmetric function of α and β.

  In order to find the value of a symmetric function of α and β, express the given function in terms of $\alpha + \beta$ and $\alpha\beta$. The following results may be useful.

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

(iii) $\alpha^4 + \beta^4 = (\alpha^3 + \beta^3)(\alpha + \beta) - \alpha\beta(\alpha^2 + \beta^2)$

(iv) $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$

(v) $|\alpha - \beta| = |\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\,|$

(vi) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

(vii) $\alpha^3 - \beta^3 = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$

(viii) $\alpha^4 - \beta^4 = (\alpha + \beta)(\alpha - \beta)(\alpha^2 + \beta^2)$

CONDITION FOR COMMON ROOTS

- **Only one root is common :** Let α be the common root of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$.

  $\therefore \quad a_1\alpha^2 + b_1\alpha + c_1 = 0,\ a_2\alpha^2 + b_2\alpha + c_2 = 0$

  By Cramer's rule :

  $$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

  or $$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

  $$\Rightarrow \quad \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2},\ \alpha \neq 0$$

  $\therefore$ The condition for only one root common is $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

- **Both roots are common :** Then required condition is $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

PROPERTIES OF QUADRATIC EQUATION

- If $f(a)$ and $f(b)$ are of opposite signs then at least one or in general odd number of roots of the equation $f(x) = 0$ lie between a and b.
- If $f(a) = f(b)$ then there exists a point c between a and b such that $f'(c) = 0,\ a < c < b$.
- If α is a root of the equation $f(x) = 0$ then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$, or $(x - \alpha)$ is factor of $f(x)$.
- If the roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio $\left(i.e. \dfrac{\alpha_1}{\beta_1} = \dfrac{\alpha_2}{\beta_2}\right)$ then $\dfrac{b_1^2}{b_2^2} = \dfrac{a_1c_1}{a_2c_2}$.

POSITION OF ROOTS

- If $f(x) = 0$ is an equation and a, b are two real numbers such that $f(a) \cdot f(b) < 0$, then at least one real root or an odd number of real roots of $f(x) = 0$ lie between a and b. In case $f(a)$ and $f(b)$ are of the same sign, then either no real root or an even number of real roots of $f(x) = 0$ lie between a and b.
- Every equation of an odd degree has at least one real root, whose sign is opposite to that of its last term, provided the coefficient of the first term is +ve. For example, $x^3 - 3x + 2 = 0$ has one real negative root.
- Every equation of an even degree whose last term (constant term) is –ve and the coefficient of first term +ve has at least two real roots, one +ve and one –ve.
- If an equation has only one change of sign, it has one +ve root and no more.
- If all the terms of an equation are +ve and the equation involves no odd power of x, then all its roots are complex.

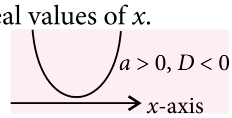
THE QUADRATIC EXPRESSION

Let $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c,\ a \neq 0$ be a quadratic expression. Since,

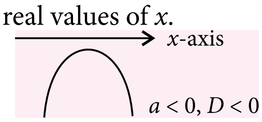
$$f(x) = a\left[\left\{x + \frac{b}{2a}\right\}^2 - \left\{\frac{b^2 - 4ac}{4a^2}\right\}\right]$$

For some values of x, $f(x)$ may be positive, negative or zero. This gives the following cases :

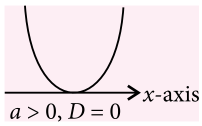
- $a > 0$ and $D < 0$, so $f(x) > 0$ for all $x \in R$ *i.e.*, $f(x)$ is positive for all real values of x.



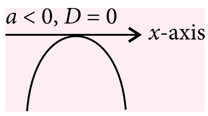
- $a < 0$ and $D < 0$, so $f(x) < 0$ for all $x \in R$ *i.e.*, $f(x)$ is negative for all real values of x.



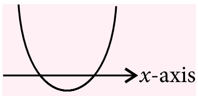
- $a > 0$ and $D = 0$, so $f(x) \geq 0$ for all $x \in R$ *i.e.*, $f(x)$ is positive for all real values of x except at vertex, where $f(x) = 0$.



- $a < 0$ and $D = 0$, so $f(x) \leq 0$ for all $x \in R$ *i.e.* $f(x)$ is negative for all real values of x except at vertex, where $f(x) = 0$.



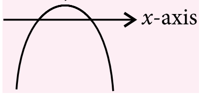
- $a > 0$ and $D > 0$, let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$, then $f(x) > 0$ for all $x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.



$a > 0, D > 0$

- $a < 0$ and $D > 0$, let $f(x) = 0$ have two real roots α and β ($\alpha < \beta$), then $f(x) < 0$ for all $x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$ for all $x \in (\alpha, \beta)$.

$a < 0, D > 0$



- If $a > 0$ (< 0), then $f(x)$ has a minimum (maximum) value at $x = -\dfrac{b}{2a}$ and this value is given by $[f(x)]_{\min(\max)} = \dfrac{4ac - b^2}{4a}$

DESCARTE'S RULE OF SIGNS

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in $f(x)$.

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in $f(-x)$.

RATIONAL ALGEBRAIC INEQUATIONS

- Values of rational expression $P(x)/Q(x)$ for real values of x, where $P(x)$ and $Q(x)$ are quadratic expressions : To find the values attained by rational expression of the form $\dfrac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$ for real values of x, the following algorithm will explain the procedure :
  - **Algorithm**
  - Equate the given rational expression to y.
  - Obtain a quadratic equation in x by simplifying the expression in step-I.
  - Obtain the discriminant of the quadratic equation in step-II.
  - Put discriminant ≥ 0 and solve the inequation for y. The values of y so obtained determines the set of values attained by the given rational expression.
- **Solution of rational algebraic inequation:** If $P(x)$ and $Q(x)$ are polynomial in x, then the inequation $\dfrac{P(x)}{Q(x)} > 0, \dfrac{P(x)}{Q(x)} < 0, \dfrac{P(x)}{Q(x)} \geq 0$ and $\dfrac{P(x)}{Q(x)} \leq 0$ are known as inequations,

where $\dfrac{P(x)}{Q(x)} = \dfrac{(x-a_1)^{p_1}(x-a_2)^{p_2}\ldots(x-a_k)^{p_k}}{(x-b_1)^{m_1}(x-b_2)^{m^2}\ldots(x-b_t)^{m_t}}$

such that $a_i \neq b_j$ and $a_i < b_j$

To solve these inequations we use the wavy curve methods as explained in the following algorithm.

- **Algorithm**
- Obtain $P(x)$ and $Q(x)$.
- Factorize $P(x)$ and $Q(x)$ into linear factors.

(A) Say $f(x) = \dfrac{g(x)}{p(x)}$

$$= \frac{(x-a_1)^{p_1}(x-a_2)^{p_2}\ldots(x-a_k)^{p_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_t)^{m_t}} > 0, < 0, \geq 0, \leq 0,$$

where $p(x) \neq 0$

(i) where a_i, b_j are real numbers $\begin{pmatrix} i = 1, 2, 3 \ldots\ldots, k \\ j = 1, 2, 3 \ldots\ldots, t \end{pmatrix}$ such that $a_i \neq b_j$ and $a_i < b_j$

(ii) $p_1, p_2 \ldots\ldots p_k, m_1, m_2, \ldots\ldots, m_t$ are natural numbers

(iii) $a_1 < a_2 < a_3 \ldots\ldots < a_k$ and $b_1 < b_2 < b_3 \ldots\ldots < b_t$

- Make the coefficient of x positive in all factors.
- Obtain critical points by equating all factors to zero.
- Plot the critical points on the number line. If there are n critical points, they divide the number line into $(n + 1)$ regions.
- Now after marking of the numbers on the number line, put positive sign (+ ve) in the right of the biggest of these number (Here, take b_t as biggest as we assume $a_i < b_j$) so put down + ve sign in the right of b_t.
- If m^t is even, we put down positive sign in the left of b^t and if m^t is odd then, we put down negative (minus) sign in the left of b^t and for the next cases, we proceed as following rule.
- When passing through the next point the polynomial changes its sign if its power is odd and polynomial have the same sign if its power is even and continue the process by the same rule.
- The solution of $f(x) > 0$ is the union of all those intervals in which we have put +ve sign.
- The solution of $f(x) < 0$ is the union of all those intervals in which we have put negative (minus) sign.

- **Lagrange's identity**
  If $a_1, a_2, a_3, b_1, b_2, b_3 \in R$ then
  $$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$$
  $$=(a_1b_2-a_2b_1)^2+(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2$$

EQUATIONS WHICH CAN BE REDUCED TO LINEAR, QUADRATIC AND BIQUADRATIC EQUATIONS

- An equation of the form $(x - a)(x - b)(x - c)(x - d) = A$, where $a < b < c < d$, $b - a = d - c$, can be solved by a change of variable.
  i.e., $y=\dfrac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$
  or $y=x-\dfrac{(a+b+c+d)}{4}$
- An equation of the form $(x - a)(x - b)(x - c)(x - d) = Ax^2$ where $ab = cd$, can be reduced to a collection of two quadratic equations by a change of variable *i.e.*, $y=x+\dfrac{ab}{x}$.
- An equation of the form $(x - a)^4 + (x - b)^4 = A$ can also be solved by a change of variable, *i.e.*, making a substitution $y=\dfrac{(x-a)+(x-b)}{2}$.

SOME IMPORTANT RESULTS

- For the quadratic equation $ax^2 + bx + c = 0$.
  - One root will be reciprocal of the other if $a = c$.
  - One root is zero if $c = 0$.
  - Roots are equal in magnitude but opposite in sign if $b = 0$.
  - Both roots are zero if $b = c = 0$.
  - Roots are positive if a and c are of the same sign and b is of the opposite sign.
  - Roots are of opposite sign if a and c are of opposite signs.
  - Roots are negative if a, b, c are of the same sign.
- Let $f(x) = ax^2 + bx + c = 0$, where $a > 0$. Then
  - Conditions for both the roots of $f(x) = 0$ to be greater than a given number k are $b^2-4ac>0; f(k)>0; \dfrac{-b}{2a}>k$.
  - Conditions for both the roots of $f(x) = 0$ to be less than a given number k are $b^2 - 4ac > 0$; $f(k) > 0$; $\dfrac{-b}{2a}<k$.
  - The number k lies between the roots of $f(x) = 0$, if $b^2 - 4ac > 0$; $f(k) < 0$.
  - Conditions for exactly one root of $f(x) = 0$ to lie between k_1 and k_2 is $f(k_1)\, f(k_2) < 0$, $b^2 - 4ac > 0$.
  - Conditions for both the roots of $f(x) = 0$ are confined between k_1 and k_2 is $f(k_1) > 0$ and $f(k_2)>0\ \forall a, b^2-4ac>0$ and $k_1<\dfrac{-b}{2a}<k_2$ where $k_1 < k_2$.
  - Conditions for both the numbers k_1 and k_2 lie between the roots of $f(x) = 0$ is $b^2 - 4ac > 0$; $f(k_1) < 0$; $f(k_2) < 0$.
- If α is a repeated root of the quadratic equation $f(x) = ax^2 + bx + c = 0$. Then α is also a root of the equation $f'(x) = 0$.
- In the equation $ax^2 + bx + c = 0$ $[a, b, c \in R]$ if $a + b + c = 0$ then the roots are 1, c/a and if $a - b + c = 0$, then the roots are -1 and $-c/a$.
- If the ratio of roots of the quadratic equation $ax^2\ bx+c=0\ (a\neq 0)$ be $p:q$, then $pqb^2=(p+q)^2ac$.
- If the roots of the equation $ax^2 + bx + c = 0$ are α, β then the roots of $cx^2 + bx + a = 0$ will be $1/\alpha$, $1/\beta$.
- The roots of the equation $ax^2 + bx + c = 0$ are reciprocal to $a'x^2 + b'x + c' = 0$ if $(cc'-aa')^2=(ba'-cb')(ab'-bc')$.
- If an equation has only one change of sign, it has one +ve root and no more.
- If all the terms of an equation are +ve and the equation involves no odd power of x, then its all roots are complex.

PROBLEMS

Single Correct Answer Type

1. The number of roots of the quadratic equation $8\sec^2\theta - 6\sec\theta + 1 = 0$ is
   (a) Infinite (b) 1
   (c) 2 (d) 0
2. The number which exceeds its positive square root by 12 is
   (a) 9 (b) 16
   (c) 25 (d) none of these
3. If $x^2 + y^2 = 25$, $xy = 12$, then $x =$
   (a) $\{3, 4\}$ (b) $\{3, -3\}$
   (c) $\{3, 4, -3, -4\}$ (d) $\{-3, -3\}$

4. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ are
(a) 1 (b) 2 (c) 3 (d) 4

5. The number of real roots of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ are
(a) 1 (b) 2
(c) infinite (d) doesn't exist

6. The number of real solutions of the equation $|x^2 + 4x + 3| + 2x + 5 = 0$ are
(a) 1 (b) 2 (c) 3 (d) 4

7. The value of $2+\cfrac{1}{2+\cfrac{1}{2+\ldots\ldots\ldots\infty}}$ is
(a) $1-\sqrt{2}$ (b) $1+\sqrt{2}$
(c) $1\pm\sqrt{2}$ (d) none of these

8. The number of solutions of $\dfrac{\log 5+\log(x^2+1)}{\log(x-2)}=2$ is
(a) 2 (b) 3 (c) 1 (d) 0

9. If $x=\sqrt{6+\sqrt{6+\sqrt{6+\ldots\text{to }\infty}}}$, then
(a) x is an irrational number
(b) $2 < x < 3$ (c) $x = 3$
(d) none of these

10. A real root of the equation $\log_4\{\log_2(\sqrt{x+8}-\sqrt{x})\}=0$ is
(a) 1 (b) 2 (c) 3 (d) 4

11. The roots of $|x - 2|^2 + |x - 2| - 6 = 0$ are
(a) 0, 4 (b) –1, 3 (c) 4, 2 (d) 5, 1

12. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$ where $ac \neq 0$, then $P(x) \cdot Q(x) = 0$ has at least
(a) four real roots (b) two real roots
(c) four imaginary roots
(d) none of these

13. Both the roots of the given equation :
$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ are
(a) positive (b) negative
(c) real (d) imaginary

14. If the roots of the equations $px^2 + 2qx + r = 0$ and $qx^2-2\left(\sqrt{pr}\right)x+q=0$ be real, then
(a) $p = q$ (b) $q^2 = pr$
(c) $p^2 = qr$ (d) $r^2 = pq$

15. The equation $x^{(3/4)(\log_2 x)^2+(\log_2 x)-5/4}=\sqrt{2}$ has
(a) at least one real solution
(b) exactly three real solutions
(c) exactly one irrational solution
(d) all the these

16. If $a > 0$, $b > 0$, $c > 0$ then both the roots of the equation $ax^2 + bx + c = 0$
(a) are real and negative
(b) have negative real parts
(c) are rational numbers
(d) none of these

17. If the roots of the equation $x^2 - 8x + (a^2 - 6a) = 0$ are real, then
(a) $-2 < a < 8$ (b) $2 < a < 8$
(c) $-2 \le a \le 8$ (d) $2 \le a \le 8$

18. If $x^2 + 2x + 2xy + my - 3$ has two rational factors, then the value of m will be
(a) –6, –2 (b) –6, 2
(c) 6, –2 (d) 6, 2

19. Roots of the equations $2x^2 - 5x + 1 = 0$, $x^2 + 5x + 2 = 0$ are
(a) reciprocal and of same sign
(b) reciprocal and of opposite sign
(c) equal in product (d) none of these

20. The expression $y = ax^2 + bx + c$ has always the same sign as c if
(a) $4ac < b^2$ (b) $4ac > b^2$
(c) $ac < b^2$ (d) $ac > b^2$

21. The roots of the equation $(a^2 + b^2)t^2 - 2(ac + bd)t + (c^2 + d^2) = 0$ are equal, then
(a) $ab = dc$ (b) $ac = bd$
(c) $ad + bc = 0$ (d) $\dfrac{a}{b}=\dfrac{c}{d}$

22. If $k \in (-\infty, -2) \cup (2, \infty)$, then the roots of the equation $x^2 + 2kx + 4 = 0$ are
(a) complex (b) real and unequal
(c) real and equal
(d) one real and one imaginary

23. If the equation $(m - n)x^2 + (n - l)x + l - m = 0$ has equal roots, then l, m and n satisfy
(a) $2l = m + n$ (b) $2m = n + l$
(c) $m = n + l$ (d) $l = m + n$

24. $x^2 + x + 1 + 2k(x^2 - x - 1) = 0$ is a perfect square for how many values of k ?
(a) 2 (b) 0 (c) 1 (d) 3

25. The values of 'a' and 'b' for which equation $x^4 - 4x^3 + ax^2 + bx + 1 = 0$ have four positive real roots are

(a) $-6, -4$ (b) $-6, 5$ (c) $-6, 4$ (d) $6, -4$

26. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is

(a) $acx^2 + (a + c)bx + (a + c)^2 = 0$
(b) $abx^2 + (a + c)bx + (a + c)^2 = 0$
(c) $acx^2 + (a + b)cx + (a + c)^2 = 0$
(d) none of these

27. If a root of the equation $ax^2 + bx + c = 0$ be reciprocal of a root of the equation then $a'x^2 + b'x + c' = 0$, then

(a) $(cc' - aa')^2 = (ba' - cb')(ab' - bc')$
(b) $(bb' - aa')^2 = (ca' - bc')(ab' - bc')$
(c) $(cc' - aa')^2 = (ba' + cb')(ab' + bc')$
(d) none of these

28. If α and β be the roots of the equation $2x^2 + 2(a + b)x + a^2 + b^2 = 0$, then the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

(a) $x^2 - 2abx - (a^2 - b^2)^2 = 0$
(b) $x^2 - 4abx - (a^2 - b^2)^2 = 0$
(c) $x^2 - 4abx + (a^2 - b^2)^2 = 0$
(d) none of these

29. If α, β be the roots of the equation $2x^2 - 2(m^2 + 1)x + m^4 + m^2 + 1 = 0$, then $\alpha^2 + \beta^2 =$

(a) 0 (b) 1 (c) m (d) m^2

30. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then $\frac{\alpha}{a\beta + b} + \frac{\beta}{a\alpha + b} =$

(a) $2/a$ (b) $2/b$ (c) $2/c$ (d) $-2/a$

31. If α, β are the roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, then $q^2 - p^2 =$

(a) $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$
(b) $(\alpha + \gamma)(\beta + \gamma)(\alpha - \delta)(\beta + \delta)$
(c) $(\alpha + \gamma)(\beta + \gamma)(\alpha + \delta)(\beta + \delta)$
(d) None of these

32. If α, β be the roots of $x^2 - px + q = 0$ and α', β' be the roots of $x^2 - p'x + q' = 0$, then the value of $(\alpha - \alpha')^2 + (\beta - \alpha')^2 + (\alpha - \beta')^2 + (\beta - \beta')^2$ is

(a) $2\{p^2 - 2q + p'^2 - 2q' - pp'\}$
(b) $2\{p^2 - 2q + p'^2 - 2q' - qq'\}$
(c) $2\{p^2 - 2q - p'^2 - 2q' - pp'\}$
(d) $2\{p^2 - 2q - p'^2 - 2q' - qq'\}$

Multiple Correct Answer Type

33. If α and β are the roots of the equation $x^2 + px + q = 0$, and α^4 and β^4 are the roots of $x^2 - rx + s = 0$, the roots of $x^2 - 4qx + 2q^2 - r = 0$ are always

(a) both real
(b) both positive
(c) both negative
(d) one positive and one negative

34. In a ΔABC $\tan A$ and $\tan B$ satisfy the inequation $\sqrt{3}x^2 - 4x + \sqrt{3} < 0$. Then

(a) $a^2 + b^2 - ab < c^2$ (b) $a^2 + b^2 > c^2$
(c) $a^2 + b^2 + ab > c^2$ (d) all of the above

35. If $b^2 \geq 4ac$ for the equation $ax^4 + bx^2 + c = 0$, then all the roots of the equation will be real if

(a) $b > 0, a < 0, c > 0$ (b) $b < 0, a > 0, c > 0$
(c) $b < 0, a > 0, c < 0$ (d) $b > 0, a < 0, c < 0$

36. Let x, y, z be positive reals. Then

(a) $\frac{4}{x} + \frac{9}{y} + \frac{16}{z} \geq 81$ if $x + y + z = 1$

(b) $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$

(c) If $xyz = 1$, then $(1 + x)(1 + y)(1 + z) < 8$

(d) If $x + y + z = 1$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$

Comprehension Type

Paragraph for Q. No. 37 to 39

Consider the equation : $\sin^2 x + a\sin x + b = 0, x \in (0, \pi)$

37. The above equation has exactly two roots and both are equal then

(a) $a = 1$ (b) $a = -1$
(c) $b = 1$ (d) $b = -1$

38. The above equation has exactly three distinct solutions then

(a) $b \in (-1, 0)$ (b) $b \in (0, 1)$
(c) $b \in [-1, 0]$ (d) $b \in [0, 1]$

39. The above equation has four solutions then which of the following are not true?

(a) $a \in (-2, 0)$ (b) $b \in (0, 1)$
(c) $a^2 - 4b > 0$ (d) $b \in (-1, 0)$

Matrix - Match Type

40. Match the following.

	Column-I		Column-II
A.	If ${}^8C_{k+2} + 2 \cdot {}^8C_{k+3} + {}^8C_{k+4} > {}^{10}C_4$, then quadratic equations whose roots are α, β and α^k, β^k have m common roots, then m =	p.	1
B.	If the number of solutions of the equation $\lvert 2x^2 - 5x + 3\rvert + (x - 1) = 0$ is (are) n, then n =	q.	2
C.	If the constant term of the quadratic expression $\sum_{k=1}^{n}\left(x-\frac{1}{k+1}\right)\left(x-\frac{1}{k}\right)$ as $n \to \infty$ is p, then p =	r.	0
C.	The equation $x^2 + 4a^2 = 1 - 4ax$ and $x^2 + 4b^2 = 1 - 4bx$ have only one root in common, then the value of $\lvert a - b\rvert$ is	s.	–1

41. Match the following.

	Column-I		Column-II
A.	$f(x) = x^2 + 2x + 8$	p.	positive integral roots
B.	$f(x) = -x^2 + 4x - 1$	q.	min $(f(x)) = 7$
C.	$f(x) = x^2 + 6x + 5$	r.	max $(f(x)) = 3$
D.	$f(x) = x^3 - 6x^2 + 11x - 6$	s.	negative integral roots

Integer Answer Type

42. Largest integral value of m for which the quadratic expression $y = x^2 + (2m + 6)x + 4m + 12$ is always positive, $\forall\, x \in R$, is

43. The number of the distinct real roots of the equation $(x + 1)^5 = 2(x^5 + 1)$ is

44. Let α, β be the roots of $x^2 - x + p = 0$ and λ, δ be the roots of $x^2 - 4x + q = 0$ such that $\alpha, \beta, \gamma, \delta$ are in G.P and $p \geq 2$. If $a, b, c \in \{1, 2, 3, 4, 5\}$, let the number of equation of the form $ax^2 + bx + c = 0$ which have real roots be r, then the minimum value of $\frac{pqr}{1536} =$

45. Number of positive integer n for which $n^2 + 96$ is a perfect square is

SOLUTIONS

1. (d) : $8\sec^2\theta - 6\sec\theta + 1 = 0$

$\Rightarrow\ \sec\theta = \frac{1}{2}$ or $\sec\theta = \frac{1}{4}$

But $\sec\theta \geq 1$ or $\sec\theta \leq -1$.

Hence the given equation has no solution.

2. (b) : Let the required number be x

So, $x = \sqrt{x} + 12 \Rightarrow x - 12 = \sqrt{x}$

$\Rightarrow\ x^2 - 25x + 144 = 0$

$\Rightarrow\ x^2 - 16x - 9x + 144 = 0 \quad \Rightarrow \quad x = 16$

Since $x = 9$ does not hold the condition.

3. (c) : We have, $x^2 + y^2 = 25$ and $xy = 12$

$\Rightarrow\ x^2 + \left(\frac{12}{x}\right)^2 = 25 \ \Rightarrow\ x^4 + 144 - 25x^2 = 0$

$\Rightarrow\ (x^2 - 16)(x^2 - 9) = 0 \ \Rightarrow\ x^2 = 16$ and $x^2 = 9$

$\Rightarrow\ x = \pm 4$ and $x = \pm 3$

4. (d) : Given : $|x|^2 - 3|x| + 2 = 0$

Case I : $x < 0$. This gives $x^2 + 3x + 2 = 0$

$\Rightarrow\ (x + 2)(x + 1) = 0 \ \Rightarrow\ x = -2, -1$

Also $x = -1, -2$ satisfy $x < 0$

So $x = -1, -2$ is solution in this case.

Case II : $x > 0$. This gives $x^2 - 3x + 2 = 0$

$\Rightarrow\ (x - 2)(x - 1) = 0 \ \Rightarrow\ x = 2, 1$

So $x = 2, 1$ is solution in this case.

Hence the number of solutions are four *i.e.* $x = -1, 1, 2, -2$.

5. (d) : Given equation, $e^{\sin x} - e^{-\sin x} - 4 = 0$

Let $e^{\sin x} = y$, then given equation can be written as

$$y^2 - 4y - 1 = 0 \ \Rightarrow\ y = 2 \pm \sqrt{5}$$

But the value of $y = e^{\sin x}$ is always positive, so

$$y = 2 + \sqrt{5} \quad (\because 2 < \sqrt{5})$$

$\Rightarrow\ \log_e y = \log_e(2+\sqrt{5}) \ \Rightarrow\ \sin x = \log_e(2+\sqrt{5}) > 1$

which is impossible, since $\sin x$ cannot be greater than 1. Hence we cannot find any real value of x which satisfies the given equation.

6. (b) : We have, $|x^2 + 4x + 3| + 2x + 5 = 0$

Case I : $x^2 + 4x + 3 > 0$

This gives $x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow\ x^2 + 6x + 8 = 0 \ \Rightarrow\ (x + 2)(x + 4) = 0$

$\Rightarrow\ x = -2, -4$

$x = -2$ is not satisfying the condition $x^2 + 4x + 3 > 0$,

So $x = -4$ is the only solution of the given equation.

Case II : $x^2 + 4x + 3 < 0$

This gives $-(x^2 + 4x + 3) + 2x + 5 = 0$

$\Rightarrow\ -x^2 - 2x + 2 = 0 \ \Rightarrow\ x^2 + 2x - 2 = 0$

$\Rightarrow \quad x=-1+\sqrt{3},-1-\sqrt{3}$

Hence, $x=-(1+\sqrt{3})$ satisfy the given condition $x^2 + 4x + 3 < 0$, while $x=-1+\sqrt{3}$ is not satisfying the condition. Thus number of real solutions are two.

7. (b) : Let $x=2+\cfrac{1}{2+\cfrac{1}{2+.....\infty}}$

$\Rightarrow \quad x=2+\frac{1}{x} \quad \Rightarrow \quad x=1\pm\sqrt{2}$

But the value of the given expression cannot be negative or less than 2, therefore $1+\sqrt{2}$ is required answer.

8. (d) : We have, $\frac{\log 5+\log(x^2+1)}{\log(x-2)}=2$

$\Rightarrow \log\{5(x^2 + 1)\} = \log(x - 2)^2$

$\Rightarrow 5(x^2 + 1) = (x - 2)^2 \Rightarrow 4x^2+4x+1=0 \Rightarrow x=-\frac{1}{2}$

But for $x=-\frac{1}{2}, \log(x-2)$ is not meaningful.

Hence it has no root.

9. (c) : We have, $x=\sqrt{6+\sqrt{6+\sqrt{6+\ldots \text{to } \infty}}}$

i.e., $x=\sqrt{6+x}, x>0 \Rightarrow x^2=6+x, x>0$

$\Rightarrow x^2 - x - 6 = 0, x > 0 \quad \Rightarrow \quad x = 3$ as $x > 0$

10. (a) : $\log_4\left\{\log_2(\sqrt{x+8}-\sqrt{x})\right\}=0$

$\Rightarrow \quad 4^0=\log_2\left(\sqrt{x+8}-\sqrt{x}\right) \Rightarrow 2^1=\sqrt{x+8}-\sqrt{x}$

$\Rightarrow \quad 4=x+8+x-2\sqrt{x^2+8x}$

$\Rightarrow \quad 2\sqrt{x^2+8x}=2x+4 \quad \Rightarrow \quad x^2+8x=x^2+4+4x$

$\Rightarrow \quad 4x = 4 \quad \Rightarrow \quad x = 1$

11. (a) : When $x < 2$, $(x - 2)^2 - (x - 2) - 6 = 0$

$\Rightarrow x^2 - 4x + 4 - x + 2 - 6 = 0 \quad \Rightarrow \quad x^2 - 5x = 0$

$\Rightarrow x(x - 5) = 0 \quad \Rightarrow \quad x = 0$

When $x \geq 2$; $(x - 2)^2 + (x - 2) - 6 = 0$

$\Rightarrow x^2 - 4x + 4 + x - 2 - 6 = 0$

$\Rightarrow x^2 - 3x - 4 = 0 \quad \Rightarrow \quad (x - 4)(x + 1) = 0 \Rightarrow x = 4$

12. (b) : Let all four roots are imaginary. Then roots of both equations $P(x) = 0$ and $Q(x) = 0$ are imaginary. Thus $b^2 - 4ac < 0$, $d^2 + 4ac < 0$. So $b^2 + d^2 < 0$, which is impossible unless $b = 0$, $d = 0$.

So, if $b \neq 0$ or $d \neq 0$, then at least two roots must be real.

If $b = 0$, $d = 0$, we have the equations

$P(x) = ax^2 + c = 0$ and $Q(x) = -ax^2 + c = 0$

or $\quad x^2=-\frac{c}{a}; x^2=\frac{c}{a}$.

As one of $\frac{c}{a}$ and $-\frac{c}{a}$ must be positive, so two roots must be real.

13. (c) : Given equation,

$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

can be re-written as

$3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0$

$D = 4\{(a + b + c)^2 - 3(ab + bc + ca)\}$ $(\because b^2 - 4ac = D)$

$= 4(a^2 + b^2 + c^2 - ab - bc - ac)$

$= 2\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \geq 0$

Hence, both roots are always real.

14. (b) : Given equations $px^2 + 2qx + r = 0$ ...(i)

and $qx^2-2(\sqrt{pr})x+q=0$ ...(ii)

have real roots, then from (i) we get

$4q^2 - 4pr \geq 0 \Rightarrow q^2 - pr \geq 0 \Rightarrow q^2 \geq pr$ ...(iii)

and from (ii), we get $4(pr) - 4q^2 \geq 0$ (for real root)

$\Rightarrow pr \geq q^2$ ...(iv)

From (iii) and (iv), we get result $q^2 = pr$

15. (d) : For $x > 0$, the given equation can be written as $\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}=\log_x \sqrt{2}=\frac{1}{2}\log_x 2$

$\Rightarrow \quad \frac{3}{4}t^2+t-\frac{5}{4}=\frac{1}{2}\left(\frac{1}{t}\right)$

$\left[\text{By putting } t = \log_2 x \Rightarrow \log_x 2=\frac{1}{t}\right]$

$\Rightarrow 3t^3 + 4t^2 - 5t - 2 = 0 \Rightarrow (t - 1)(t + 2)(3t + 1) = 0$

$\Rightarrow \quad \log_2 x=t=1,-2,-\frac{1}{3}$

$\Rightarrow \quad x=2,2^{-2},2^{-1/3}$ or $x=2,\frac{1}{4},\frac{1}{2^{1/3}}$

16. (b) : The roots of the equations are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(i) Let $b^2 - 4ac > 0$, $b > 0$

Now if $a > 0$, $c > 0$, $b^2 - 4ac < b^2$

$\Rightarrow$ The roots are negative.

(ii) Let $b^2 - 4ac < 0$, then the roots are given by

$$x=\frac{-b\pm i\sqrt{(4ac-b^2)}}{2a}, \quad (i=\sqrt{-1}) \qquad (\because b > 0)$$

which are imaginary and have negative real part.

$\therefore$ In each case, the roots have negative real part.

17. (c) : Roots of $x^2 - 8x + (a^2 - 6a) = 0$ are real.

So $D \geq 0$

$\Rightarrow 64 - 4(a^2 - 6a) \geq 0 \Rightarrow 16 - a^2 + 6a \geq 0$

$\Rightarrow a^2 - 6a - 16 \leq 0 \Rightarrow (a - 8)(a + 2) \leq 0$

Now we have two cases:
Case I : $(a - 8) \le 0$ and $(a + 2) \ge 0$
$\Rightarrow$ $a \le 8$ and $a \ge -2$
Case II : $(a - 8) \ge 0$ and $(a + 2) \le 0$
$\Rightarrow$ $a \ge 8$ and $a \le -2$ but it is impossible
Therefore, we get $-2 \le a \le 8$

18. (c) : Given expression is $x^2 + 2x + 2xy + my - 3$ or $x^2 + 2x(1 + y) + (my - 3)$
But factors are rational, so $b^2 - 4ac$ is a perfect square.
Now, $4\{(1 + y)^2 - (my - 3)\} > 0$
$\Rightarrow$ $4\{y^2 + 1 + 2y - my + 3\} > 0$
$\Rightarrow$ $y^2 + 2y - my + 4 > 0$
Hence $2y - my = \pm 4y$ {as it is perfect square}
$\Rightarrow$ $2y - my = 4y$ $\Rightarrow$ $m = -2$ or $2y - my = -4y \Rightarrow m = 6$

19. (b) : We have that, if roots of $ax^2 + bx + c = 0$ are α, β, then the roots of $cx^2 + bx + a = 0$ will be $\frac{1}{\alpha}, \frac{1}{\beta}$ and hence roots of $cx^2 - bx + a = 0$ will be $-\frac{1}{\alpha}, -\frac{1}{\beta}$.

20. (b) : Let $f(x) = ax^2 + bx + c$. Then $f(0) = c$. Thus the graph of $y = f(x)$ meets y-axis at $(0, c)$.
If $c > 0$, then by hypothesis $f(x) > 0$. This means that the curve $y = f(x)$ does not meet x-axis.
If $c < 0$, then by hypothesis $f(x) < 0$, which means that the curve $y = f(x)$ is always below x-axis and so it does not intersect with x-axis. Thus in both cases $y = f(x)$ does not intersect with x-axis *i.e.* $f(x) \ne 0$ for any real x. Hence $f(x) = 0$ *i.e.*, $ax^2 + bx + c = 0$ has imaginary roots and so $b^2 < 4ac$.

21. (d) : Accordingly, $\{2(ac + bd)\}^2 = 4(a^2 + b^2)(c^2 + d^2)$
$\Rightarrow$ $4a^2c^2 + 4b^2d^2 + 8abcd = 4a^2c^2 + 4a^2d^2 + 4b^2c^2 + 4b^2d^2$
$\Rightarrow$ $4a^2d^2 + 4b^2c^2 - 8abcd = 0$ $\Rightarrow$ $4(ad - bc)^2 = 0$
$\Rightarrow$ $ad = bc \Rightarrow \frac{a}{b} = \frac{c}{d}$

22. (b) : Given equation is $x^2 + 2kx + 4 = 0$
Put $k = -3$, $x^2 - 6x + 4 = 0$ $\Rightarrow$ $x = 3 + \sqrt{5}, 3 - \sqrt{5}$
Put $k = 3$, $x^2 + 6x + 4 = 0$ $\Rightarrow$ $x = -3 + \sqrt{5}, -3 - \sqrt{5}$
i.e., Roots are real and unequal.

23. (b) : Since, roots are equal. So, $b^2 - 4ac = 0$
$\Rightarrow$ $(n - l)^2 - 4(m - n)(l - m) = 0$
$\Rightarrow$ $n^2 + l^2 - 2nl - 4(ml - nl - m^2 + mn) = 0$
$\Rightarrow$ $l^2 + n^2 + (2m)^2 + 2nl - 4mn - 4ml = 0$
$\Rightarrow$ $(l + n - 2m)^2 = 0$ $\Rightarrow$ $l + n = 2m$

24. (a) : Given equation
$(1 + 2k)x^2 + (1 - 2k)x + (1 - 2k) = 0$
If equation is a perfect square then roots are equal
i.e., $(1 - 2k)^2 - 4(1 + 2k)(1 - 2k) = 0$ *i.e.*, $k = \frac{1}{2}, \frac{-3}{10}$.
Hence total number of values = 2.

25. (d) : Let four real roots are $\alpha, \beta, \gamma, \delta$, then equation is $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) = 0$
$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma)x^2 - (\alpha\beta\gamma + \beta\gamma\delta + \alpha\beta\delta + \alpha\gamma\delta)x + \alpha\beta\gamma\delta = 0$$
$$x^4 - \Sigma\alpha \cdot x^3 + \Sigma\alpha\beta \cdot x^2 - \Sigma\alpha\beta\gamma \cdot x + \alpha\beta\gamma\delta = 0$$
On comparing with $x^4 - 4x^3 + ax^2 + bx + 1 = 0$, we get
$\Sigma\alpha = 4$, $\Sigma\alpha\beta = a$, $\Sigma\alpha\beta\gamma = -b$, $\alpha\beta\gamma\delta = 1$
$\therefore$ $\alpha = \beta = \gamma = \delta = 1$
Now, $\Sigma\alpha\beta = a$
$\therefore$ $a = \alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma = 6$
$-b = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 4$

26. (a) : Here, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$
If roots are $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$, then sum of roots are
$$= \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta} = -\frac{b}{ac}(a + c)$$
and product $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$
$$= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta} = 2 + \frac{c}{a} + \frac{a}{c} = \frac{2ac + c^2 + a^2}{ac} = \frac{(a + c)^2}{ac}$$
Hence, required equation is given by
$$x^2 + \frac{b}{ac}(a + c)x + \frac{(a + c)^2}{ac} = 0$$
$\Rightarrow$ $acx^2 + (a + c)bx + (a + c)^2 = 0$

27. (a) : Let α be a root of first equation, then $1/\alpha$ be a root of second equation.
Therefore, $a\alpha^2 + b\alpha + c = 0$ and $a'\frac{1}{\alpha^2} + b'\frac{1}{\alpha} + c' = 0$
or $c'\alpha^2 + b'\alpha + a' = 0$
Hence, $\frac{\alpha^2}{ba' - b'c} = \frac{\alpha}{cc' - aa'} = \frac{1}{ab' - bc'}$
$\Rightarrow$ $(cc' - aa')^2 = (ba' - cb')(ab' - bc')$

28. (b) : Sum of roots, $\alpha + \beta = -(a + b)$ and
Product of roots, $\alpha\beta = \frac{a^2 + b^2}{2}$
$\Rightarrow$ $(\alpha + \beta)^2 = (a + b)^2$ and $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$
$= 2ab - (a^2 + b^2) = -(a - b)^2$
Now the required equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

$$x^2-\{(\alpha+\beta)^2+(\alpha-\beta)^2\}x+(\alpha+\beta)^2(\alpha-\beta)^2=0$$
$$\Rightarrow\ x^2-\{(a+b)^2-(a-b)^2\}x-(a+b)^2(a-b)^2=0$$
$$\Rightarrow\ x^2-4abx-(a^2-b^2)^2=0$$

29. (d) : We have, α and β are the roots of $2x^2-2(m^2+1)x+m^4+m^2+1=0$

$$\Rightarrow\ \alpha+\beta=\frac{2(m^2+1)}{2}=m^2+1 \qquad \text{.... (i)}$$

$$\text{and } \alpha\beta=\frac{m^4+m^2+1}{2} \qquad \text{.... (ii)}$$

Therefore, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=(m^2+1)^2-2\frac{(m^4+m^2+1)}{2}$$

$=m^4+2m^2+1-m^4-m^2-1=m^2$

30. (d) : From given equation, we have

$$\alpha+\beta=-\frac{b}{a},\alpha\beta=\frac{c}{a} \text{ and } \alpha^2+\beta^2=\frac{(b^2-2ac)}{a^2}$$

$$\text{Now, } \frac{\alpha}{a\beta+b}+\frac{\beta}{a\alpha+b}=\frac{\alpha(a\alpha+b)+\beta(a\beta+b)}{(a\beta+b)(a\alpha+b)}$$

$$=\frac{a(\alpha^2+\beta^2)+b(\alpha+\beta)}{\alpha\beta a^2+ab(\alpha+\beta)+b^2}=\frac{a\frac{(b^2-2ac)}{a^2}+b\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)a^2+ab\left(-\frac{b}{a}\right)+b^2}$$

$$=\frac{b^2-2ac-b^2}{a^2c-ab^2+ab^2}=\frac{-2ac}{a^2c}=-\frac{2}{a}$$

31. (a) : As given, $\alpha+\beta=-p$, $\alpha\beta=1$, $\gamma+\delta=-q$ and $\gamma\delta=1$

Now, $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$

$$=\{\alpha\beta-\gamma(\alpha+\beta)+\gamma^2\}\{\alpha\beta+\delta(\alpha+\beta)+\delta^2\}$$
$$=(1+p\gamma+\gamma^2)(1-p\delta+\delta^2)=(p\gamma-q\gamma)(-p\delta-q\delta)$$
$$=-\gamma\delta(p-q)(p+q)=q^2-p^2$$

32. (a) : As given,

$$\alpha+\beta=p,\ \alpha\beta=q,\ \alpha'+\beta'=p',\ \alpha'\beta'=q'$$

Now, $(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$

$$=2(\alpha^2+\beta^2)+2(\alpha'^2+\beta'^2)-2\alpha'(\alpha+\beta)-2\beta'(a+\beta)$$
$$=2\left\{(\alpha+\beta)^2-2\alpha\beta+(\alpha'+\beta')^2-2\alpha'\beta'-(\alpha+\beta)(\alpha'+\beta')\right\}$$
$$=2\{p^2-2q+p'^2-2q'-pp'\}$$

33. (a, d) : We have $\alpha+\beta=-p$, $\alpha\beta=q$, $\alpha^4+\beta^4=r$ and $\alpha^4\beta^4=s$

Therefore, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q$, so that

$$r=\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=(p^2-2q)^2-2q^2$$

i.e., $(p^2)^2-4q(p^2)+2q^2-r=0$

This shows that p^2 is one root of $x^2-4qx+2q^2-r=0$. If its other root is γ, we have $\gamma+p^2=4q$, *i.e.*, $\gamma=4q-p^2$. Further the discriminant of this quadratic equation is $(4q)^2-4(2q^2-r)=8q^2+4[(p^2-2q)^2-2q^2]$

$$=4(p^2-2q^2)\ge 0$$

So that both roots, p^2 and $-p^2+4q$ are real. Since α and β are real $p^2-4q\ge 0$, *i.e.*, $-p^2+4q\le 0$. Thus the roots of $x^2-4qx+2q^2-r=0$ are positive and negative.

34. (a, c) : $(x-\sqrt{3})(x\sqrt{3}-1)<0$

$$\Rightarrow\ x \text{ lies between } \frac{1}{\sqrt{3}} \text{ and } \sqrt{3}$$

$$\Rightarrow\ \text{Both } \tan A \text{ and } \tan B \text{ lie between } \frac{1}{\sqrt{3}} \text{ and } \sqrt{3}.$$

$\Rightarrow$ Both A and B lie between 30° and 60°.

$\Rightarrow$ $60° < C < 120°$

$$\Rightarrow\ -\frac{1}{2}<\frac{a^2+b^2-c^2}{2ab}<\frac{1}{2}$$

35. (b, d) : Put $x^2=t$, $t\ge 0$, we get

$at^2+bt+c=0$, $t\ge 0$

$$\text{Sum of roots}=-\frac{b}{a}>0,\ \text{Product of roots}=\frac{c}{a}>0$$

36. (a, b, d) : (a) $x+y+z=1$

$$\Rightarrow\ \frac{4}{x}+\frac{9}{y}+\frac{16}{z}=\left(\frac{4}{x}+\frac{9}{y}+\frac{16}{z}\right)(x+y+z)$$

$$=29+\left(\frac{4y}{x}+\frac{9x}{y}\right)+\left(\frac{16y}{z}+\frac{9z}{y}\right)+\left(\frac{4z}{x}+\frac{16x}{z}\right)$$

(b) Use A.M. $\ge$ G.M.

$$\frac{(y+z)+(z+x)+(x+y)}{3}\ge\sqrt[3]{(y+z)(z+x)(x+y)}$$

$$\therefore\ \frac{2}{3}(x+y+z)\ge\sqrt[3]{(y+z)(z+x)(x+y)} \qquad \text{... (i)}$$

Similarly,

$$\frac{\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}}{3}\ge[(y+z)(z+x)(x+y)]^{-1/3} \qquad \text{...(ii)}$$

(d) On multiplication of (i) and (ii) and expanding, we get the desired result.

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge 3^2$$

37. (c) : If the given equation have two equal roots, both should be equal to $\pi/2$

$$\therefore\ \sin x=1.\ \text{Product of roots}=\frac{b}{1}=1\ \Rightarrow b=1$$

38. (b) : If the given equation have three solutions, one root should definitely be $\pi/2$. $\therefore$ $\sin x_1=1$

Now, we should get two more roots and thus $\sin x \in (0, 1)$ $\therefore$ Product of roots $= b = \sin x_1 \cdot \sin x$ $= 1 \cdot \sin x = \sin x$ $\therefore$ $b \in (0, 1)$

39. (d)

40. A-q; B-p; C-p; D-p

(A) Given, ${}^8C_{k+2} + 2\,{}^8C_{k+3} + {}^8C_{k+4} > {}^{10}C_4$

$\Rightarrow$ $({}^8C_{k+2} + {}^8C_{k+3}) + ({}^8C_{k+3} + {}^8C_{k+4}) > {}^{10}C_4$

$\Rightarrow$ ${}^9C_{k+3} + {}^9C_{k+4} > {}^{10}C_4$

$\Rightarrow$ ${}^{10}C_{k+4} > {}^{10}C_4$.

Only ${}^{10}C_5 > {}^{10}C_4$ $\Rightarrow$ $k + 4 = 5$ $\Rightarrow$ $k = 1$

$\therefore$ $\alpha^k = \alpha$ and $\beta^k = \beta$

Hence quadratic equation having roots α and β and α^k and β^k are identical and have both roots common.

$\therefore$ $m = 2$

(B) For $1 \le x < \frac{3}{2}$ or $\frac{3}{2} \le x < \infty$, $x - 1 > 0$

Therefore no solution is possible

For $x \le 1$ given equation is

$(2x^2 - 5x + 3) + x - 1 = 0$

$\Rightarrow$ $2x^2 - 4x + 2 = 0$ $\Rightarrow$ $x^2 - 2x + 1 = 0$ $\Rightarrow$ $x = 1$

$\therefore$ The equation has only one solution

$\therefore$ $n = 1$

(C) Constant term, $C = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$

$$t_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore\quad C = \sum_{r=1}^{n} t_r = 1 - \frac{1}{n+1} \qquad \therefore\ p = 1$$

(D) Given, $(x + 2a)^2 = 1$ and $(x + 2b)^2 = 1$

$x = \pm 1 - 2a$ and $x = \pm 1 - 2b$

$1 - 2a = -1 - 2b$ $\Rightarrow$ $b - a = -1$

$-1 - 2a = 1 - 2b$ $\Rightarrow$ $b - a = 1$

$\Rightarrow$ $|a - b| = 1$

41. A – q; B – r; C – s; D – p

(A) Coefficient of x^2 is + ve

$$\min(f(x)) = \frac{4ac - b^2}{4a} = 7$$

(B) Coefficient x^2 is –ve

$$\max(f(x)) = \frac{4ac - b^2}{4a} = 3$$

(C) Roots of given equation are –5, –1

(D) The roots are 1, 2, 3, only +ve integral roots.

42. (0) : $D < 0$ $\Rightarrow$ $-3 < m < 1$ $\Rightarrow$ $m = 0$

43. (3) : $(x + 1)^5 = 2(x^5 + 1)$

Let $f(x) = \frac{(x+1)^5}{(x^5+1)}$ $\qquad (x \ne -1)$

$\Rightarrow$ $f'(x) = \frac{5(x+1)^4(1-x^4)}{(x^5+1)^2}$ $\Rightarrow$ $x = 1$ is maximum

As, $f(0) = 1$ and $f(1) = 16$

And $\lim_{x\to\pm\infty} f(x) = 1$ $\Rightarrow$ $f(x) = 2$ has two solutions but given equation has three solutions because $x = -1$ included.

44. (1) : $(\alpha + \beta) = 1$, $\alpha\beta = p$, $\gamma + \delta = 4$, $\gamma\delta = q$

Since $\alpha, \beta, \gamma, \delta$ are in G.P.

$$\therefore\quad \frac{\beta}{\alpha} = \frac{\delta}{\gamma} \quad\Rightarrow\quad \frac{\beta+\alpha}{\beta-\alpha} = \frac{\delta+\gamma}{\delta-\gamma}$$

$$\Rightarrow\quad \frac{(\beta+\alpha)^2}{(\beta+\alpha)^2 - 4\alpha\beta} = \frac{(\delta+\gamma)^2}{(\delta+\gamma)^2 - 4\delta\gamma}$$

$$\Rightarrow\quad \frac{1}{1-4p} = \frac{16}{16-4q} = \frac{4}{4-q} \quad\Rightarrow\quad 4 - q = 4 - 16p$$

Now, $p \ge 2$. ...(i) $\therefore$ $q \ge 32$ ...(ii)

For the given equation $ax^2 + bx + c = 0$ to have real roots $b^2 - 4ac \ge 0$.

$$\therefore\quad ac \le \frac{b^2}{4}$$

b	$\frac{b^2}{4}$	Possible values of ac such that $ac \le \frac{b^2}{4}$	No. of possible pairs (a, c)
2	1	1	1
3	2.25	1, 2	3
4	4	1, 2, 3, 4	8
5	6.25	1, 2, 3, 4, 5, 6	12
		Total	24

Hence number of quadratic equation with real roots, $r = 24$ …(iii)

Now from (i), (ii) and (iii) the minimum value of

$pqr = 2 \times 32 \times 24 = 1536$

45. (4) : Suppose m is positive integer such that $n^2 + 96 = m^2$ then

$(m - n)(m + n) = 96$

As $m - n < m + n$ and $m - n$, $m + n$ both must be even

So, the only possibilities are

$m - n = 2$, $m + n = 48$; $m - n = 4$, $m + n = 24$

$m - n = 6$, $m + n = 16$; $m - n = 8$, $m + n = 12$

So, the solutions of (m, n) are

(25, 23), (14, 10), (11, 5), (10, 2).

CLASS XI Series 4

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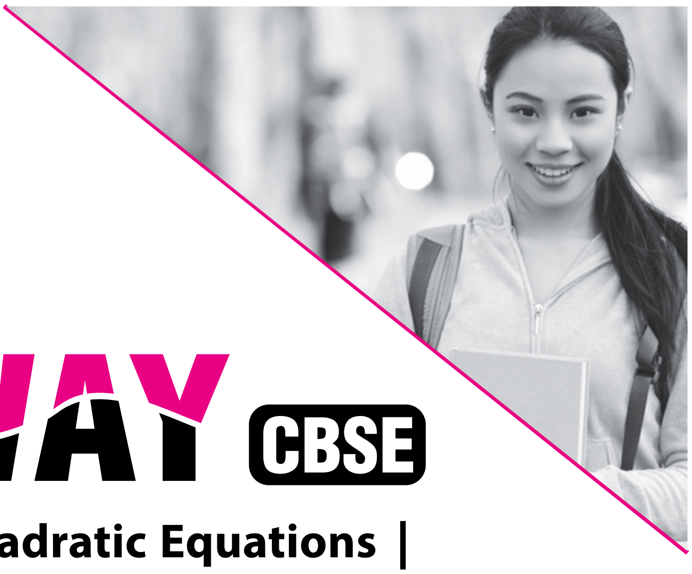


Complex Numbers and Quadratic Equations | Linear Inequalities

IMPORTANT FORMULAE

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

- *The system of number $c = \{z; z = a + ib; a, b \in \mathbf{R}\}$ is called the set of complex number.*
- *Two complex numbers z_1 and z_2 are said to be equal, iff $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$*
- *Conjugate of z, $\bar{z} = a - ib$*
- *Properties of Conjugate*

If z, z_1, z_2 are complex numbers, then

- $\overline{(\bar{z})} = z$
- $z + \bar{z} = 2Re(z)$
- $z - \bar{z} = 2i\,Im(z)$
- $z = \bar{z} \Leftrightarrow z$ *is purely real*
- $z + \bar{z} = 0 \Leftrightarrow z$ *is purely imaginary*
- $z\bar{z} = \{Re(z)\}^2 + \{Im(z)\}^2$
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$
- *Modulus of z,* $|z| = \sqrt{a^2 + b^2}$
- *Properties of Modulus*
  If $z, z_1, z_2 \in C$, then
- $|z| = 0 \Leftrightarrow z = 0$ *i.e.* $Re(z) = Im(z) = 0$
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $-|z| \leq Re(z) \leq |z|$; $-|z| \leq Im(z) \leq |z|$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \neq 0$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2)$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$
  $= (a^2 + b^2)(|z_1|^2 + |z_2|^2), \forall\, a,b \in R$
- Square root of z,

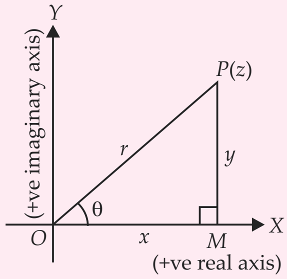
$$\sqrt{z} = \sqrt{a+ib} = \begin{cases} \pm\left[\sqrt{\dfrac{|z|+a}{2}} + i\sqrt{\dfrac{|z|-a}{2}}\right] & \text{for } b > 0 \\ \pm\left[\sqrt{\dfrac{|z|+a}{2}} - i\sqrt{\dfrac{|z|-a}{2}}\right] & \text{for } b < 0 \end{cases}$$

- *Multiplicative inverse of $z = (a + ib)$; $a, b \neq 0$ is*

$$\frac{1}{z} = z^{-1} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

▶ *Argand Plane and Polar Representation*

In complex plane, point P can be represented by the ordered pair (r, θ) where r is called modulus of z and is denoted as mod(z) or |z| while θ is called the argument (or amplitude) of z given by



$\theta = arg(z) = tan^{-1}\left(\frac{b}{a}\right) = tan^{-1}\left(\frac{Im(z)}{Re(z)}\right)$

Also, the value of θ such that $-\pi < \theta \le \pi$ is called the principal argument of z and $z = r(\cos\theta + i\sin\theta)$ is known as polar form or trigonometrical form.

▶ *Properties of Argument of Complex Numbers*

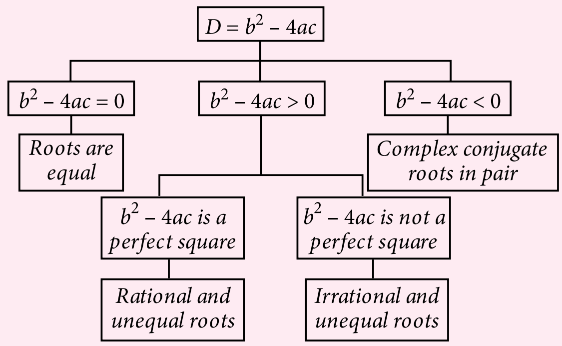
- $arg(z_1z_2) = arg(z_1) + arg(z_2)$
- $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$
- $arg\left(\frac{z}{\bar{z}}\right) = 2\,arg(z)$
- $arg(z^n) = n\,arg(z) + 2n\pi,\ n \in I$
- If $arg\left(\frac{z_2}{z_1}\right) = \theta$, then $arg\left(\frac{z_1}{z_2}\right) = 2n\pi - \theta, n \in I$
- $arg(z_1\bar{z}_2) = arg(z_1) - arg(z_2)$
- $arg(\bar{z}) = -arg(z)$
- $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow arg(z_1) - arg(z_2) = \frac{\pi}{2}$
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow arg(z_1) = arg(z_2)$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary

▶ *Triangle Inequality*

- $|z_1 + z_2| \le |z_1| + |z_2|$
- $|z_1 - z_2| \ge |z_1| - |z_2|$

▶ *Solution of quadratic equation $ax^2 + bx + c = 0$ is*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



▶ ***Some Results on Roots of an Equation***

- *An equation of degree n has n roots, real or imaginary.*
- *An odd degree equation has atleast one real root, whose sign is opposite to that of its last term provided that the coefficient of highest degree term is positive.*
- *Every equation of even degree whose constant term is negative and the coefficient of highest degree term is positive, has atleast two real roots, one positive and one negative.*

LINEAR INEQUALITIES

▶ $]a, b[$ or $(a, b) = \{x \in R : a < x < b\}$

▶ $[a, b] = \{x \in R : a \le x \le b\}$

▶ $]a, b]$ or $(a, b] = \{x \in R : a < x \le b\}$

▶ $[a, b) = \{x \in R : a \le x < b\}$

▶ $[a, \infty[$ or $[a, \infty) = \{x \in R : a \le x\}$

▶ $]a, \infty[$ or $(a, \infty) = \{x \in R : a < x\}$

▶ $]-\infty, a]$ or $(-\infty, a] = \{x \in R : x \le a\}$

▶ $]-\infty, a[$ or $(-\infty, a) = \{x \in R : x < a\}$

▶ $]-\infty, \infty[$ or $(-\infty, \infty) = \{x \in R\}$

▶ *If α, β are real numbers and α < β, then*

- $(x - \alpha)(x - \beta) > 0 \Leftrightarrow x < \alpha$ or $x > \beta$
- $(x - \alpha)(x - \beta) \ge 0 \Leftrightarrow x \le \alpha$ or $x \ge \beta$
- $(x - \alpha)(x - \beta) < 0 \Leftrightarrow \alpha < x < \beta$
- $(x - \alpha)(x - \beta) \le 0 \Leftrightarrow \alpha \le x \le \beta$
- $(x - \alpha)^2 \ge 0,\ \forall\, x \in R$
- $\frac{x-\alpha}{x-\beta}, x \ne \beta$ and $\frac{x-\beta}{x-\alpha}, x \ne \alpha$ have same sign scheme as that of $(x - \alpha)(x - \beta)$.

▶ $|x| < a \Leftrightarrow -a < x < a$

▶ $|x| > a \Leftrightarrow x < -a$ or $x > a$

▶ *If a and b are +ve real numbers and a < b, then*

- $a < |x| < b \Leftrightarrow -b < x < -a$ or $a < x < b$
- $a \le |x| \le b \Leftrightarrow -b \le x \le -a$ or $a \le x \le b$
- $|a + b| \le |a| + |b| \Leftrightarrow ab \ge 0;\ \forall\, a, b \in R$
- $|a - b| \ge ||a| - |b|| \Leftrightarrow ab \ge 0;\ \forall\, a, b \in R$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Solve : $|x + 1| - |x - 1| = 2$
2. Solve the equation : $2x^2 + 3 = 0$.
3. For what real values of x and y, are the following numbers $x^2 - 7x + 9yi$ and $y^2\, i + 20i - 12$ equal ?
4. Given $x \in \{-3, -4, -5, -6\}$ and $9 \le 1 - 2x$, find the possible values of x. Also represent its solution set on the number line.
5. Express the following in the standard form $a + ib$:
   $(2i - i^2)^2 + (1 - 3i)^3$

SHORT ANSWER TYPE

6. Solve the equation $2x^2 + 3ix + 2 = 0$ using the general expression for a quadratic equation.
7. If z is a complex number and $iz^3 + z^2 - z + i = 0$, then prove that $|z| = 1$.
8. If $x \in W$, find the solution set of $\frac{3}{5}x - \frac{2x-1}{3} > 1$.
9. Solve the inequation : $|4 - x| + 1 < 3$.
10. If $z = x + iy$ and $\frac{|z-1-i|+4}{3|z-1-i|-2} = 1$, show that $x^2 + y^2 - 2x - 2y = 7$.

LONG ANSWER TYPE - I

11. Show that if $\left|\frac{z-5i}{z+5i}\right| = 1$, then z is a real number.
12. Find the square roots of the complex number $-7 - 24i$.
13. Find two complex numbers such that their sum is 6 and the product is 14.
14. Convert the complex number $3\left(\cos\frac{5\pi}{3} - i\sin\frac{\pi}{6}\right)$ into polar form.
15. Solve the following inequality $2y - 3 < y + 2 \le 3y + 5$, $y \in R$ and represent the solution set on the number line.

LONG ANSWER TYPE - II

16. Solve the following system of linear inequalities graphically : $x - 2y \le 3$, $3x + 4y > 12$, $x \ge 0$, $y \ge 1$
17. If $(1 + a)z = b + ic$ and $a^2 + b^2 + c^2 = 1$, $a, b, c \in R$, then show that $\frac{1+iz}{1-iz} = \frac{a+ib}{1+c}$.
18. Solve the equation $x^2 + (\sqrt{3} - 2\sqrt{2}i)x - 2\sqrt{6}i = 0$ using the general expression for a quadratic equation.
19. Solve the inequation : $|x - 1| + |x - 2| + |x - 3| \le 6$.
20. Solve the equation : $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$

SOLUTIONS

1. Given, $|x + 1| - |x - 1| = 2$

$\Rightarrow |x + 1| - |x - 1| = (x + 1) - (x - 1)$

$\Rightarrow |x + 1| - |x - 1| \le |(x + 1) - (x - 1)|$

$[\because |a| - |b| \le |a - b| \text{ iff } ab \ge 0]$

$\Rightarrow x \le -1$ or $x \ge 1$

$\therefore$ Solution set $= (-\infty, -1] \cup [1, \infty)$

2. We have, $2x^2 + 3 = 0$

$\Rightarrow (\sqrt{2}x)^2 - (\sqrt{3}i)^2 = 0 \Rightarrow (\sqrt{2}x + \sqrt{3}i)(\sqrt{2}x - \sqrt{3}i) = 0$

$\Rightarrow x = \pm\sqrt{\frac{3}{2}}\, i$

3. Given $x^2 - 7x + 9yi = y^2 i + 20i - 12$

$\Rightarrow (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$

$\Rightarrow x^2 - 7x = -12$ and $9y = y^2 + 20$

$\Rightarrow x^2 - 7x + 12 = 0$ and $y^2 - 9y + 20 = 0$

$\Rightarrow (x - 4)(x - 3) = 0$ and $(y - 5)(y - 4) = 0$

$\Rightarrow x = 4, 3$ and $y = 5, 4$.

Hence, the required values of x and y are

$x = 4, y = 5$; $x = 4, y = 4$; $x = 3, y = 5$; $x = 3, y = 4$

4. Given, $9 \le 1 - 2x$

$\Rightarrow 2x + 9 \le 1 - 2x + 2x$

$\Rightarrow 2x + 9 \le 1$

$\Rightarrow 2x + 9 + (-9) \le 1 + (-9)$

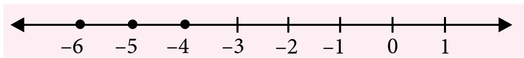
$\Rightarrow 2x \le -8$

$\Rightarrow x \le -4$

But $x \in \{-3, -4, -5, -6\}$,

$\therefore$ The solution set is $\{-4, -5, -6\}$.

The solution set is shown by thick dots on the number line.



5. Given, $(2i - i^2)^2 + (1 - 3i)^3$

$= (2i + 1)^2 + (1 - 3i)^3$

$= (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3)$

$= -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i$

6. We have, $2x^2 + 3ix + 2 = 0$

Comparing with the general form $ax^2 + bx + c = 0$, we get

$a = 2$, $b = 3i$ and $c = 2$

Now, $D = b^2 - 4ac = -9 - 16 = -25 = 25i^2$

$\therefore \sqrt{D} = \pm 5i$

Hence, the roots for the equation are

$$x = \frac{-3i - 5i}{4} = -2i \text{ or } x = \frac{-3i + 5i}{4} = \frac{i}{2}$$

7. Given, $iz^3 + z^2 - z + i = 0$

$\Rightarrow iz^3 + z^2 + i^2 z + i = 0 \quad (\because i^2 = -1)$

$\Rightarrow z^2(iz + 1) + i(iz + 1) = 0$

$\Rightarrow (z^2 + i)(iz + 1) = 0$

$\Rightarrow\ z^2 = -i$ or $iz = -1 \Rightarrow z = -\frac{1}{i} = i$

Now $z = i \Rightarrow |z| = |i| = \sqrt{0^2 + 1^2} = 1$

and $z^2 = -i \Rightarrow |z^2| = |-i| = 1$

$\Rightarrow\ |z| = 1$

8. Given, $\frac{3}{5}x - \frac{2x-1}{3} > 1$

$\Rightarrow\ 9x - 5(2x - 1) > 15$

$\Rightarrow\ 9x - 10x + 5 > 15$

$\Rightarrow\ -x > 15 - 5 \Rightarrow -x > 10$

$\Rightarrow\ x < -10$ (Multiplying by -1)

But $x \in W \Rightarrow$ The solution set $= \phi$

9. Given, $|4 - x| + 1 < 3$

$\Rightarrow\ |4 - x| < 2$

$\Rightarrow\ -2 < 4 - x < 2$ $[\because |x| < k \Leftrightarrow -k < x < k]$

$\Rightarrow\ -2 - 4 < -x < 2 - 4$

$\Rightarrow\ -6 < -x < -2$

$\Rightarrow\ 6 > x > 2$

$\Rightarrow\ 2 < x < 6$

10. Given, $\frac{|z-1-i|+4}{3|z-1-i|-2} = 1$

$\Rightarrow\ 3|z - 1 - i| - 2 = |z - 1 - i| + 4$

$\Rightarrow\ 2|z - 1 - i| = 6 \Rightarrow |x + iy - 1 - i| = 3$

$\Rightarrow\ |(x - 1) + i(y - 1)| = 3$

$\Rightarrow\ \sqrt{(x-1)^2 + (y-1)^2} = 3$

$\Rightarrow\ x^2 - 2x + 1 + y^2 - 2y + 1 = 9$

$\Rightarrow\ x^2 + y^2 - 2x - 2y = 7$

11. Given, $\left|\frac{z-5i}{z+5i}\right| = 1 \Rightarrow \frac{|z-5i|}{|z+5i|} = 1$

$\Rightarrow\ |z - 5i| = |z + 5i|$

$\Rightarrow\ |x + iy - 5i| = |x + iy + 5i|$

$\Rightarrow\ |x + i(y - 5)| = |x + i(y + 5)|$

$\Rightarrow\ \sqrt{x^2 + (y-5)^2} = \sqrt{x^2 + (y+5)^2}$

$\Rightarrow\ x^2 + y^2 - 10y + 25 = x^2 + y^2 + 10y + 25$

$\Rightarrow\ -20y = 0 \Rightarrow y = 0$

$\Rightarrow\ z = x + 0i \Rightarrow z$ is a real number.

12. Let $\sqrt{-7-24i} = x + iy$, where x and y are real numbers.

On squaring both sides, we get

$-7 - 24i = (x^2 - y^2) + i(2xy)$

Equating real and imaginary parts, we get

$x^2 - y^2 = -7$ ...(i) and $2xy = -24$ ...(ii)

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow\ (x^2 + y^2)^2 = (-7)^2 + (-24)^2$ (Using (i) and (ii))

$\Rightarrow\ (x^2 + y^2)^2 = 49 + 576 = 625$

$\Rightarrow\ x^2 + y^2 = \pm 25$

As x and y are real numbers, $x^2 + y^2$ cannot be negative

$\therefore\ x^2 + y^2 = 25$ ...(iii)

On adding (i) and (iii), we get

$2x^2 = 18 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

From (ii), $y = -\frac{12}{(\pm 3)} = \pm 4$

$\therefore\ \sqrt{-7-24i} = x + iy = 3 - 4i$ or $-3 + 4i$

13. Let the two numbers be α and β, then according to question, $\alpha + \beta = 6$ ...(i) and $\alpha\beta = 14$ ...(ii)

Substituting the value of β from (i) in (ii), we get

$\alpha(6 - \alpha) = 14$ ...(iii)

$\Rightarrow\ 6\alpha - \alpha^2 - 14 = 0 \Rightarrow \alpha^2 - 6\alpha + 14 = 0$

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 1, b = -6, c = 14$

Discriminant, $D = b^2 - 4ac = (-6)^2 - 4 \times 1 \times 14 = -20$

$\therefore\ \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm 2\sqrt{5}i}{2} = 3 \pm \sqrt{5}i$

From (i), When $\alpha = 3 + \sqrt{5}i$, $\beta = 6 - (3 + \sqrt{5}i) = 3 - \sqrt{5}i$;

when $\alpha = 3 - \sqrt{5}i$, $\beta = 6 - (3 - \sqrt{5}i) = 3 + \sqrt{5}i$

Hence, the two numbers are $3 + \sqrt{5}i$ and $3 - \sqrt{5}i$.

14. We have, $\cos\frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore\ 3\left(\cos\frac{5\pi}{3} - i\sin\frac{\pi}{6}\right) = 3\left(\frac{1}{2} - i\frac{1}{2}\right) = \frac{3}{2} - \frac{3}{2}i$

Let $z = \frac{3}{2} - \frac{3}{2}i = r(\cos\theta + i\sin\theta)$

Then, $r\cos\theta = \frac{3}{2}$ and $r\sin\theta = -\frac{3}{2}$

On squaring and adding, we get

$$r^2(\cos^2\theta + \sin^2\theta) = \left(\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2$$

$\Rightarrow\ r^2 = \frac{9}{4} + \frac{9}{4} = \frac{9}{2} \Rightarrow r = \frac{3}{\sqrt{2}}$

$\therefore\ \cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$

The values of θ such that $-\pi < \theta \le \pi$ and satisfying both the above equation is given by $\theta = -\frac{\pi}{4}$.

Hence,

$$3\left(\cos\frac{5\pi}{3} - i\sin\frac{\pi}{6}\right) = \frac{3}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right),$$

which is the required polar form.

15. Given, $2y - 3 < y + 2 \le 3y + 5,\ y \in R$

$\Rightarrow\ 2y - 3 < y + 2$ and $y + 2 \le 3y + 5$

$\Rightarrow\ 2y < y + 5$ and $y \le 3y + 3$

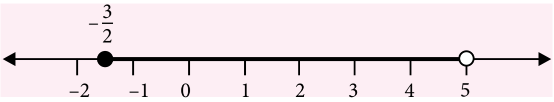
$\Rightarrow\ y < 5$ and $-2y \le 3$

$\Rightarrow\ y < 5$ and $y \ge -\dfrac{3}{2}$

$\Rightarrow\ -\dfrac{3}{2} \le y < 5$

$\therefore$ The solution set is $\left\{y : y \in R, -\dfrac{3}{2} \le y < 5\right\}$ *i.e.* $\left[-\dfrac{3}{2}, 5\right)$

The solution set is shown by the thick portion on the number line.

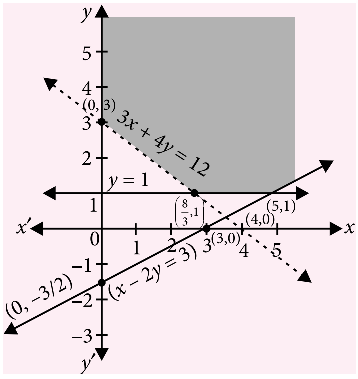


16. Converting the given inequalities into equations,

$x - 2y = 3$ ...(i), $\quad 3x + 4y = 12$ ...(ii)

$x \ge 0$ ...(iii), $\quad y \ge 1$ ...(iv)

Now, plotting (i), (ii), (iii) and (iv) on graph,



Thus, the solution set consists of all the points in the shaded part except the points on the line segments $3x + 4y > 12$.

17. Given $(1 + a)z = b + ic \ \Rightarrow\ z = \dfrac{b+ic}{1+a}$

$$\Rightarrow \frac{iz}{1} = \frac{i(b+ic)}{1+a} \Rightarrow \frac{iz}{1} = \frac{-c+ib}{1+a}$$

Applying componendo and dividendo, we get

$$\frac{1+iz}{1-iz} = \frac{(1+a)+(-c+ib)}{(1+a)-(-c+ib)}$$

$$\Rightarrow \frac{1+iz}{1-iz} = \frac{(1-c)+(a+ib)}{(1+c)+(a-ib)}$$

$$\Rightarrow \frac{1+iz}{1-iz} = \frac{(a+ib)\left(1+\dfrac{1-c}{a+ib}\right)}{(1+c)\left(1+\dfrac{a-ib}{1+c}\right)} \qquad \ldots(i)$$

Also, we are given that $a^2 + b^2 + c^2 = 1$

$\Rightarrow\ a^2 + b^2 = 1 - c^2 \Rightarrow (a + ib)(a - ib) = (1 + c)(1 - c)$

$$\Rightarrow \frac{a-ib}{1+c} = \frac{1-c}{a+ib}$$

$$\Rightarrow 1+\frac{a-ib}{1+c} = 1+\frac{1-c}{a+ib} \qquad \ldots(ii)$$

From (i), we get

$$\frac{1+iz}{1-iz} = \frac{a+ib}{1+c} \qquad \text{(Using (ii))}$$

18. We have, $x^2 + (\sqrt{3} - 2\sqrt{2}i)x - 2\sqrt{6}i = 0$

Comparing with the general form $ax^2 + bx + c = 0$, we get

$a = 1, b = (\sqrt{3} - 2\sqrt{2}i)$ and $c = -2\sqrt{6}i$

In this case, $D = b^2 - 4ac = (\sqrt{3} - 2\sqrt{2}i)^2 - 4 \cdot 1 \cdot (-2\sqrt{6}i)$

$= 3 + 8i^2 - 4\sqrt{6}i + 8\sqrt{6}i = -5 + 4\sqrt{6}i$

Now, $\sqrt{D} = \sqrt{-5 + 4\sqrt{6}i}$

Let $x + iy = \sqrt{-5 + 4\sqrt{6}i}$

$\Rightarrow\ x^2 - y^2 + 2ixy = -5 + 4\sqrt{6}i$

$\Rightarrow\ x^2 - y^2 = -5$ ...(i) and $2xy = 4\sqrt{6}$ ...(ii)

We know that $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow\ (x^2 + y^2)^2 = 25 + 96 = 121$

$\Rightarrow\ x^2 + y^2 = 11$ ...(iii)

Adding (i) and (iii), we get

$\Rightarrow\ x^2 = 3$ and $y^2 = 8$

$\Rightarrow\ x = \pm\sqrt{3}$ and $y = \pm 2\sqrt{2}$

$\because\ 2xy = 4\sqrt{6} > 0 \ \therefore\ x$ and y are of the same sign.

Thus, $x = \sqrt{3},\ y = 2\sqrt{2}$ or $x = -\sqrt{3}$ and $y = -2\sqrt{2}$

$\therefore\ \sqrt{D} = \sqrt{3} + 2\sqrt{2}i$ or $-(\sqrt{3} + 2\sqrt{2}i)$

$\therefore$ The roots of the equation are

$$\alpha = \frac{-(\sqrt{3}-2\sqrt{2}i)-(\sqrt{3}+2\sqrt{2}i)}{2} \text{ and}$$

$$\beta = \frac{-(\sqrt{3}-2\sqrt{2}i)+(\sqrt{3}+2\sqrt{2}i)}{2}$$

$\Rightarrow\ \alpha = -\sqrt{3}$ and $\beta = 2\sqrt{2}i$

Similarly calculating root for $\sqrt{D} = -(\sqrt{3} + 2\sqrt{2}i)$, we get

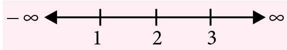
$\alpha = 2\sqrt{2}i$ and $\beta = -\sqrt{3}$

Hence the roots of the quadratic equation are $2\sqrt{2}i$ and $-\sqrt{3}$.

MPP-4 CLASS XI ANSWER KEY

1. (c)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (b)
6. (b)	**7.** (a, b)	**8.** (a,b,c,d)		**9.** (a, b)
10. (a,b,c)	**11.** (a,b,c,d)		**12.** (a, c)	**13.** (a,b,c)
14. (c)	**15.** (b)	**16.** (d)	**17.** (6)	**18.** (9)
19. (11)	**20.** (7)			

19. Here in the given inequation $|x - 1|$, $|x - 2|$ and $|x - 3|$ occur.



Now, $x - 1 = 0 \Rightarrow x = 1$,

$x - 2 = 0 \Rightarrow x = 2$

and $x - 3 = 0 \Rightarrow x = 3$

Case I. When $-\infty < x \le 1$:

$|x - 1| + |x - 2| + |x - 3| \le 6$

$\Rightarrow\ -(x - 1) - (x - 2) - (x - 3) \le 6$

$\Rightarrow\ -3x + 6 \le 6 \Rightarrow 3x \ge 0$ ...(1)

But in this case $x \le 1$ ...(2)

From (1) and (2), $0 \le x \le 1$ ...(A)

Case II. When $1 \le x \le 2$:

$|x - 1| + |x - 2| + |x - 3| \le 6$

$\Rightarrow\ (x - 1) - (x - 2) - (x - 3) \le 6$

$\Rightarrow\ -x + 4 \le 6 \Rightarrow -x \le 2 \Rightarrow x \ge -2$ ...(3)

But in this case $1 \le x \le 2$ ...(4)

From (3) and (4), $1 \le x \le 2$ ...(B)

Case III. When $2 \le x \le 3$:

$|x - 1| + |x - 2| + |x - 3| \le 6$

$\Rightarrow\ (x - 1) + (x - 2) - (x - 3) \le 6 \Rightarrow x \le 6$ ...(5)

But in this case $2 \le x \le 3$ ...(6)

From (5) and (6), $2 \le x \le 3$ ...(C)

Case IV : When $3 \le x < \infty$:

$|x - 1| + |x - 2| + |x - 3| \le 6$

$\Rightarrow\ (x - 1) + (x - 2) + (x - 3) \le 6$

$\Rightarrow\ 3x - 6 \le 6 \Rightarrow 3x \le 12 \Rightarrow x \le 4$ ...(7)

But in this case $3 \le x < \infty$ ...(8)

From (7) and (8), $3 \le x \le 4$ ...(D)

From (A), (B), (C) and (D), set of all possible real values of x is $= [0, 1] \cup [1, 2] \cup [2, 3] \cup [3, 4] = [0, 4]$

20. Let $z = x + iy$; $x, y \in R$, then $|z| = \sqrt{x^2 + y^2}$

Given : $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$

$\Rightarrow\ 2(x^2 + y^2) + (x + iy)^2 - 5 + i\sqrt{3} = 0$

$\Rightarrow\ 2x^2 + 2y^2 + x^2 - y^2 + 2ixy = 5 - i\sqrt{3}$

$\Rightarrow\ (3x^2 + y^2) + i(2xy) = 5 - i\sqrt{3}$

$\Rightarrow\ 3x^2 + y^2 = 5$ and $2xy = -\sqrt{3}$

$\Rightarrow\ 3x^2 + y^2 = 5$ ...(i) and $y = -\dfrac{\sqrt{3}}{2x}$ ...(ii)

Putting the value of y from (ii) in (i), we get

$$3x^2 + \frac{3}{4x^2} = 5 \Rightarrow 12x^4 - 20x^2 + 3 = 0$$

$\Rightarrow\ (6x^2 - 1)(2x^2 - 3) = 0 \Rightarrow x^2 = \dfrac{1}{6}, \dfrac{3}{2}$

$\Rightarrow\ x = \pm\dfrac{1}{\sqrt{6}}, \pm\sqrt{\dfrac{3}{2}}$

Using (ii), when $x = \dfrac{1}{\sqrt{6}}$, $y = -\dfrac{3}{\sqrt{2}}$;

when $x = -\dfrac{1}{\sqrt{6}}, y = \dfrac{3}{\sqrt{2}}$;

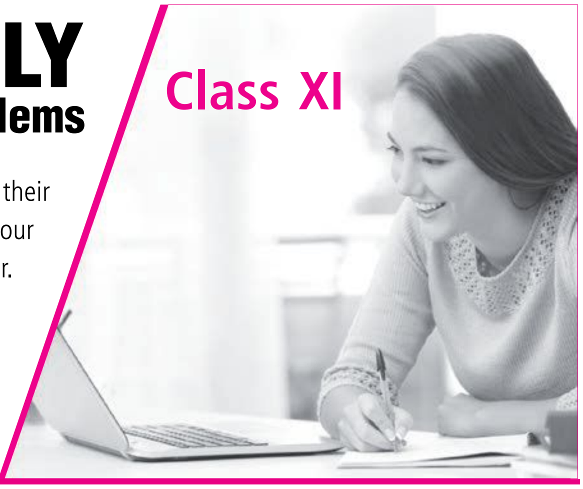
when $x = \sqrt{\dfrac{3}{2}}, y = -\dfrac{1}{\sqrt{2}}$; when $x = -\sqrt{\dfrac{3}{2}}, y = \dfrac{1}{\sqrt{2}}$

Hence the solutions are :

$$\frac{1}{\sqrt{6}} - \frac{3}{\sqrt{2}}i, -\frac{1}{\sqrt{6}} + \frac{3}{\sqrt{2}}i, \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i, -\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i$$

MPP-4 MONTHLY Practice Problems

Class XI



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Permutations & Combinations, Mathematical Induction and Mathematical Reasoning

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Negation of the statement $\sim p \rightarrow (q \vee r)$ is

(a) $p \rightarrow \sim (q \vee r)$ (b) $p \vee (q \wedge r)$

(c) $\sim p \wedge (\sim q \wedge \sim r)$ (d) $p \wedge (q \vee r)$

2. For $n \in N$, $x^{n+1} + (x+1)^{2n-1}$ is divisible by

(a) x (b) $x + 1$

(c) $x^2 + x + 1$ (d) $x^2 - x + 1$

3. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5, and 7. The smallest value of n for which this is possible is

(a) 6 (b) 7 (c) 8 (d) 9

4. The statement $p \rightarrow (q \rightarrow p)$ is equivalent to

(a) $p \rightarrow (p \wedge q)$ (b) $p \rightarrow (p \leftrightarrow q)$

(c) $p \rightarrow (p \rightarrow q)$ (d) $p \rightarrow (p \vee q)$

5. $\frac{3}{4} + \frac{15}{16} + \frac{63}{64} + \ldots$ to n terms =

(a) $n - \frac{4^n}{3} - \frac{1}{3}$ (b) $n + \frac{4^{-n}}{3} - \frac{1}{3}$

(c) $n + \frac{4^n}{3} - \frac{1}{3}$ (d) $n - \frac{4^{-n}}{3} + \frac{1}{3}$

6. The contrapositive of the statement "if $2^2 = 5$, then I get first class" is

(a) If I do not get a first class, then $2^2 = 5$

(b) If I do not get a first class, then $2^2 \neq 5$

(c) If I get a first class, then $2^2 = 5$

(d) None of these

One or More Than One Option(s) Correct Type

7. A letter lock consists of three rings marked with 15 different letters. If N denotes the number of ways in which it is possible to make unsuccessful attempts to open the lock, then

(a) $482 \mid N$

(b) N is product of 3 distinct prime numbers

(c) N is product of 4 distinct prime numbers

(d) none of these

8. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth numbers is marked (*i.e*, 1, 16, 31 etc.) This process is continued until a number is reached which has already been marked, then unmarked numbers are

(a) 200 (b) 400

(c) 600 (d) 800

9. The kindergarten teacher has 25 kids in her class. She take 5 of them at a time to zoological garden as often as she can without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by

(a) ${}^{25}C_5 - {}^{24}C_4$ (b) ${}^{24}C_5$

(c) ${}^{25}C_5 - {}^{24}C_5$ (d) ${}^{24}C_4$

10. The number of ways of selecting two 1×1 squares from a cheess board such that they

(a) have a common vertex is 98

(b) have a common side is 112

(c) neither have a common vertex nor have a common side is 1806

(d) none of these

11. $P = n(n^2 - 1)(n^2 - 4)(n^2 - 9) \ldots (n^2 - 100)$ is always divisible by $(n \in I)$

(a) $2!\ 3!\ 4!\ 5!\ 6!$ (b) $(5!)^4$
(c) $(10!)^2$ (d) $10!\ 11!$

12. If ${}^nC_{r-1} = (k^2 - 8)({}^{n+1}C_r)$, then k belongs to

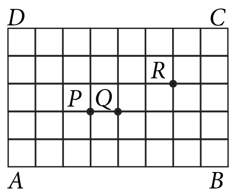
(a) $[-3, -2\sqrt{2})$ (b) $(-3, 3)$
(c) $(2\sqrt{2}, 3]$ (d) $[-2\sqrt{2}, 2\sqrt{2})$

13. Suppose a lot contains $2n$ objects of which n are identical. The number of ways to select n objects out of these $2n$ objects must be

(a) 2^n
(b) $({}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n)^{1/2}$
(c) the number of possible subsets of the set $\{a_1, a_2, \ldots, a_n\}$
(d) none of these

Comprehension Type

Consider the network of equally spaced parallel lines (6 horizontal and 9 vertical) as shown in the following figure. All small squares are of the same size. A shortest route from A to C is defined as a route consisting 8 horizontal steps (H) and 5 vertical steps (V). Since any shortest route is a typical arrangement of $8H$ and $5V$, the number of shortest routes is $\dfrac{13!}{5!8!}$. Now, answer the following questions.



14. The number of shortest routes through the junction P is

(a) 240 (b) 216 (c) 560 (d) 512

15. The number of shortest routes which pass through junction P and R is

(a) 144 (b) 240 (c) 216 (d) 256

Matrix Match Type

16. Consider all possible permutations of the letters of the word ENDEANOEL.

	Column I		Column II
P.	The number of permutations containing the word ENDEA is	1.	$5!$
Q.	The number of permutations in which the letter E occurs in the first and the last positions is	2.	$2 \times 5!$
R.	The number of permutation in which none of the letters D, L, N occurs in the last five positions is	3.	$7 \times 5!$
S.	The number of permutation in which the letters A, E, O occur only in odd positions is	4.	$21 \times 5!$

	P	Q	R	S
(a)	2	1	3	4
(b)	4	2	1	3
(c)	3	4	2	1
(d)	1	4	2	2

Integer Answer Type

17. The number of arrangements of the letters of the word FORTUNE when the order of vowels is unaltered is x, when the order of consonants is unaltered is y and when the order of vowels and consonants is unaltered is z. Then $\dfrac{x+y}{5z}$ equals

18. Let $A\ \{x_1, x_2, x_3, x_4\}$ and $B = \{y_1, y_2, y_3, y_4\}$. A function f is defined from set A to set B. Number of one-one functions such that $f(x_i) \neq y_i$ for $i = 1, 2, 3, 4$ is equal to

19. $10^{2n-1} + 1$ for all $n \in N$ is divisible by

20. If the number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is $11n$ then n equals

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT MAP

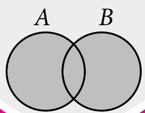
SETS

Class XI

OPERATION ON SETS

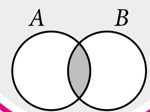
Union

$A \cup B = \{x : x \in A \text{ or } x \in B\}$



Intersection

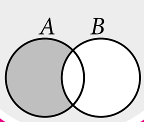
$A \cap B = \{x : x \in A \text{ and } x \in B\}$



Laws	Description
Idempotent	$A \cup A = A$ $A \cap A = A$
Commutative	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Identity	$A \cup \phi = \phi \cup A = A, A \cap \phi = \phi$ $A \cap U = U \cap A = A, A \cup U = U$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

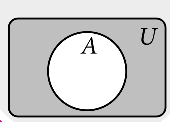
Difference

$A - B = \{x : x \in A, x \notin B\}$



Complement

$A' = U - A$



If A, B and C are finite sets, U be the finite universal set, then

1. $n(A - B) = n(A) - n(A \cap B)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. $n(A \cup B) = n(A) + n(B)$, if A, B are disjoint.
4. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
5. $n(A' \cup B') = (n(A \cap B)') = n(U) - n(A \cap B)$
6. $n(A' \cap B') = (n(A \cup B)') = n(U) - n(A \cup B)$

- **Complement Law :** $A \cup A' = U$; $A \cap A' = \phi$
- **De Morgans Law :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$
- **Law of Double complementation :** $(A')' = A$
- $\phi' = U$ and $U' = \phi$

TYPES OF SETS

Infinite set : A set whose elements cannot be listed by the natural numbers.

Equal sets : Two sets A and B are equal, if they have exactly the same elements.

Finite set : A set consisting of finite number of elements.

Singleton set : A set having exactly single element.

Empty set : A set having no element.

Equivalent sets : Two finite sets A and B are equivalent if they have same number of elements.

Universal set : A set that contains all the sets in the given context.

Subset : If each element of A is also an element of B, $A \subseteq B$.

- Every set is a subset of itself.
- Empty set is a subset of every set.
- Total number of subsets of finite set containing n elements is 2^n.

Superset : If A is a subset of B, then B is said to be superset of A, *i.e.*, $B \supseteq A$.

Power set : The set of all subsets of set A.

Proper subset : If A is a subset of B and $A \neq B$, then A is called proper subset of B *i.e.*, $A \subset B$.

SETS

A well defined collection of objects.

$x \in A$ means x is an element of set A.

Sets can be defined in two ways :

Roster Form : A set is described by characterising property $P(x)$ of its element x.

Set Builder Form : A set is described by listing elements separated by commas.

CONCEPT BOOSTERS

Class XII

MATRICES AND DETERMINANTS

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION OF MATRIX

A system of $m \times n$ numbers arranged in the form of an ordered set of m rows and n columns is called an $m \times n$ matrix. It can be read as m by n matrix. It is represented as $A = [a_{ij}]_{m \times n}$ and can be written in expanded form as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

DIFFERENT TYPES OF MATRICES

(i) **Rectangular matrix and square matrix :** If in an $m \times n$ matrix $m \neq n$, then it is a rectangular matrix
But if $m = n$, it is a square matrix.

(ii) **Column Matrix :** A matrix which has only one column and m rows is called a column matrix of length m.

(iii) **Row Matrix :** A matrix which has only one row and n columns is called a row matrix of length n.

(iv) **Diagonal Matrix :** A square matrix of any order with zero elements everywhere, except on the main diagonal, is called a diagonal matrix.

(v) **Scalar matrix :** A matrix whose diagonal elements are all equal and other entries are zero, is called a scalar matrix.

(vi) **Identity or Unit Matrix :** A square matrix in which all the elements along the main diagonal (elements of the form a_{ii}) are unity is called an identity matrix or a unit matrix. An identity matrix of order n is denoted by I_n.

(vii) **Null or Zero Matrix :** The matrix whose all elements are zero is called null matrix or zero matrix. It is usually denoted by O.

(viii) **Triangular Matrix :** A square matrix whose elements above the main diagonal or below the main diagonal are all zero is called a triangular matrix.

Note : (i) $[a_{ij}]_{n \times n}$ is said to be upper triangular matrix if $i > j \Rightarrow a_{ij} = 0$,
(ii) $[a_{ij}]_{n \times n}$ is said to be lower triangular matrix if $i < j \Rightarrow a_{ij} = 0$.

(ix) **Sub Matrix :** A matrix obtained by omitting some rows or some columns or both of a given matrix A is called a sub matrix of A.

(x) **Horizontal Matrix :** Any matrix in which the number of columns is more than the number of rows is called a horizontal matrix.

(xi) **Vertical Matrix :** Any matrix in which the number of rows is more than the number of columns is called vertical matrix.

EQUALITY OF TWO MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of the same order and their corresponding elements are equal. If two matrices A and B are equal, we write $A = B$.

OPERATION ON MATRICES

Addition of Matrices

If A and B are two matrices of the same order $m \times n$, then their sum is defined to be the matrix of order $m \times n$ obtained by adding the corresponding elements of A and B.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91).
He trains IIT and Olympiad aspirants.

Subtraction of Two Matrices

If A and B are two matrices of the same order then the sum $A + (-B)$, written as $A - B$, is the matrix obtained by subtracting B from A.

Properties of Matrix Addition

If A, B, C are three matrices of the same order, then

(i) $A + B = B + A$ (commutative law of addition)

(ii) $A + (B + C) = (A + B) + C$ (associative law of addition)

(iii) $A + O = O + A = A$

(iv) $A + B = A + C \Rightarrow B = C$

(v) $A + (-A) = (-A) + A = O$.

Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]_{m \times n}$ and λ is a scalar,

then $\lambda A = [\lambda a_{ij}]_{m \times n}$

Multiplication of Two Matrices

For two matrices A and B, the product matrix AB can be obtained if the number of columns in A = the number of rows in B.

Properties of Matrix Multiplication

(i) Matrix multiplication is associative *i.e.*, $(AB)C = A(BC)$, A, B and C are $m \times n$, $n \times p$ and $p \times q$ matrices respectively.

(ii) Multiplication of matrices is distributive over addition of matrices *i.e.*, $A(B + C) = AB + AC$

(iii) Existence of multiplicative identity of square matrices. If A is a square matrix of order n and I_n is the identity matrix of order n, then $AI_n = I_nA = A$.

(iv) Whenever AB and BA both exist, it is not necessary that $AB = BA$.

(v) The product of two matrices can be a zero matrix while neither of them is a zero matrix.

(vi) In the case of matrix multiplication of $AB = 0$, then it doesn't necessarily imply that $A = 0$ or $B = 0$ or $BA = 0$.

TRACE OF A MATRIX

Let A be a square matrix of order n. The sum of the diagonal elements of A is called the trace of A.

$$\therefore \quad \text{Trace}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}.$$

TRANSPOSE OF A MATRIX

The matrix obtained from any given matrix A, by interchanging rows and columns, is called the transpose of A and is denoted by A' or A^T.

Properties of Transpose of a matrix

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(\alpha A)' = \alpha A'$, α being any scalar.

(iv) $(AB)' = B'A'$

SPECIAL MATRICES

- **Symmetric Matrix**
  A matrix which is unchanged by transposition is called a symmetric matrix. Such a matrix is necessarily square *i.e.*, if $A = [a_{ij}]_{m \times n}$ is a symmetric matrix then $m = n$, $a_{ij} = a_{ji}$ *i.e.*, $A' = A$.
- **Skew Symmetric Matrix**
  A square matrix $A = [a_{ij}]$ is said to be skew symmetric, if $a_{ij} = -a_{ji}$ for all i and j.
  Thus if $A = [a_{ij}]_{m \times n}$ is a skew symmetric matrix, then $m = n$, $a_{ij} = -a_{ji}$ *i.e.*, $A' = -A$.
  Obviously diagonal elements of a square matrix are zero.
  Note : Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric matrix.
  i.e., $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$
  where $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$
  are symmetric and skew symmetric parts of A.
- **Orthogonal Matrix**
  A square matrix A is said to be orthogonal, if $AA' = A'A = I$, where I is a unit matrix.
  Note :
  (i) If A is orthogonal, then A' is also orthogonal.
  (ii) If A and B are orthogonal matrices then AB and BA are also orthogonal matrices.
- **Idempotent Matrix :** A square matrix A is called idempotent provided if it satisfies the relation $A^2 = A$.
- **Involutory Matrix**
  A square matrix A is called involutory matrix if $A^2 = I$
- **Periodic Matrix**
  A square matrix A is called periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be period of A. For $k = 1$, we get $A^2 = A$ and we called it to be idempotent matrix.
- **Nilpotent Matrix**
  A square matrix A is called a nilpotent matrix, if there exists a positive integer m such that

$A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A or nilpotency of matrix.

DETERMINANT

Equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ in x and y have a unique solution if and only if $a_1b_2 - a_2b_1 \neq 0$. We write $a_1b_2 - a_2b_1$ as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and call it a determinant of order 2.

Similarly the equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ have a unique solution if $a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2) \neq 0$

i.e., $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

The number a_i, b_i, c_i ($i = 1, 2, 3$) are called the elements of the determinant.

The determinant obtained by deleting the i^{th} row and j^{th} column is called the minor of the element at the i^{th} row and j^{th} column. We shall denote it by M_{ij}. The cofactor of this element is $(-1)^{i+j} M_{ij}$, denoted by C_{ij}.

Let $A = [a_{ij}]_{3 \times 3}$ be a matrix, then the corresponding determinant (denoted by det A or $|A|$) is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

It is easy to see that $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (we say that we have expanded the determinant $|A|$ along first row). Infact value of $|A|$ can be obtained by expanding it along any row or along any column. Further note that if elements of a row (column) are multiplied to the cofactors of other row (column) and then added, then the result is zero : $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$.

PROPERTIES OF DETERMINANTS

- The value of a determinant remains unaltered, if its rows are changed into columns and the columns into rows.
- If all the elements of a row (or column) of a determinant are zero, then the value of the determinant is zero.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- The interchange of any two rows (columns) of a determinant results in change of its sign.
- If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by that constant.
- If each element of a row (or column) of a determinant is a sum of two terms, then determinant can be written as sum of two determinant in the following way:

$$\begin{vmatrix} a_1 & b_1 & c_1+d_1 \\ a_2 & b_2 & c_2+d_2 \\ a_3 & b_3 & c_3+d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

- The value of a determinant remains unaltered under a column operation of the form $C_i \to C_i + \alpha C_j + \beta C_k (j, k \neq i)$ or a row operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ $(j, k \neq i)$

Product of two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 & a_1l_3+b_1m_3+c_1n_3 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 & a_2l_3+b_2m_3+c_2n_3 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 & a_3l_3+b_3m_3+c_3n_3 \end{vmatrix}$$

(row by column multiplication)

$$= \begin{vmatrix} a_1l_1+b_1l_2+c_1l_3 & a_1m_1+b_1m_2+c_1m_3 & a_1n_1+b_1n_2+c_1n_3 \\ a_2l_1+b_2l_2+c_2l_3 & a_2m_1+b_2m_2+c_2m_3 & a_2n_1+b_2n_2+c_2n_3 \\ a_3l_1+b_3l_2+c_3l_3 & a_3m_1+b_3m_2+c_3m_3 & a_3n_1+b_3n_2+c_3n_3 \end{vmatrix}$$

(row by row multiplication)

We can also multiply determinants column by row or column by column.

Limit of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

then $\lim\limits_{x \to a} \Delta(x) = \begin{vmatrix} \lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) & \lim\limits_{x\to a} h(x) \\ \lim\limits_{x\to a} l(x) & \lim\limits_{x\to a} m(x) & \lim\limits_{x\to a} n(x) \\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(x) & \lim\limits_{x\to a} w(x) \end{vmatrix}$,

provided each of nine limiting values exist finitely.

Differentiation of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

$$\text{then } \Delta'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$
$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Integration of a Determinant

$$\text{Let } \Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix},$$

where a, b, c, l, m and n are constants,

$$\text{then } \int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)\,dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

Note that if more than one row (column) of $\Delta(x)$ are variable, then in order to find $\int_a^b \Delta(x)\,dx$. First we evaluate the determinant $\Delta(x)$ by using the properties of determinants and then we integrate it.

SPECIAL DETERMINANTS

Skew symmetric Determinant

A determinant of a skew symmetric matrix of odd order is always zero.

Circulant Determinant

A determinant is called circulant if its rows (columns) are cyclic shifts of the first row (columns).

e.g., $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. It can be shown that its value is $-(a^3+b^3+c^3-3abc)$.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)\,(ab+bc+ca)$$

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]_{n \times n}$ be an $n \times n$ matrix. The transpose B' of the matrix $B = [A_{ij}]_{n \times n}$, where A_{ij} denotes the cofactor of the elements a_{ij} in the $|A|$, is called the adjoint of the matrix A and is denoted by the symbol adj A.

Thus, the adjoint of a matrix A is the transpose of the matrix formed by the cofactors of A.

Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$
(Thus $A(\text{adj } A)$ is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj }(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj }(\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj }(A^T) = (\text{adj } A)^T$

(vi) $\text{adj }(AB) = (\text{adj } B)\cdot(\text{adj } A)$

(vii) $\text{adj }(A^m) = (\text{adj } A)^m$, $m \in N$

(viii) $\text{adj }(kA) = k^{n-1}(\text{adj } A)$, $k \in R$

(ix) $\text{adj }(I_n) = I_n$

(x) $\text{adj } 0 = 0$

A is symmetric $\Rightarrow$ adj A is also symmetric.

A is diagonal $\Rightarrow$ adj A is also diagonal.

A is triangular $\Rightarrow$ adj A is also triangular.

A is singular $\Rightarrow$ $|\text{adj } A| = 0$

INVERSE OF A SQUARE MATRIX

Let A be any n–rowed square matrix. Then a matrix B, if exists, such that $AB = BA = I_n$, is called the inverse of A. Inverse of A is usually denoted by A^{-1} (if exists).

We have, $|A|I_n = A(\text{adj}A)$

$|A|\, A^{-1} = (\text{adj } A)$. Thus the necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$ and then $A^{-1} = \dfrac{\text{adj }(A)}{|A|}$.

A square matrix A is called non-singular if $|A| \neq 0$. Hence, a square matrix A is invertible if and only if A is non-singular.

Properties of Inverse of a Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^n)^{-1} = (A^{-1})^n$, $n \in \mathbb{N}$

(v) adj $(A^{-1}) = (\text{adj } A)^{-1}$

$A = \text{diag}(a_1, a_2, ..., a_n)$. A is symmetric $\Rightarrow A^{-1}$ is also symmetric ($|A| \neq 0$).

A is diagonal matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal matrix.

A is scalar matrix $\Rightarrow A^{-1}$ is also a scalar matrix.

A is triangular matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular matrix.

SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

Consider the system of linear non-homogeneous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

$$\text{Let } A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

$$\text{Let } |A| = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

obtained on replacing first column of Δ by B.

$$\text{Similarly, let } \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

It can be shown that $AX = B$, $x\Delta = \Delta_x$, $y\Delta = \Delta_y$, $z\Delta = \Delta_z$.

Determinant Method of Solution

We have the following two cases :

Case I

If $\Delta \neq 0$, then the given system of equations has unique solution, given by

$x = \Delta_x / \Delta$, $y = \Delta_y / \Delta$ and $z = \Delta_z / \Delta$.

Case II

If $\Delta = 0$, then two sub cases arise:

(a) at least one of Δ_x, Δ_y and Δ_z is non-zero, say $\Delta_x \neq 0$. Now in $x \cdot \Delta = \Delta_x$, L.H.S. is zero and R.H.S. is not equal to zero. Thus we have, no value of x satisfying $x \cdot \Delta = \Delta_x$.

Hence given system of equations has no solution.

(b) $\Delta_x = \Delta_y = \Delta_z = 0$. In this case the given equations are dependent. Delete one or two equation from the given system (as the case may be) to obtain independent equation(s). The remaining equation(s) may have no solution or infinitely many solution(s).

- **Matrix Method of Solution**

(a) $\Delta \neq 0$, then A^{-1} exists and hence $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow X = A^{-1}B$ and therefore unique values of x, y and z are obtained.

(b) We have $AX = B \Rightarrow ((\text{adj } A)A)X = (\text{adj } A)B$

$\Rightarrow \Delta X = (\text{adj } A)B$.

If $\Delta = 0$, then $\Delta X = O_{3 \times 1}$, zero matrix of order 3×1. Now if $(\text{adj } A)B = 0$, then the system $AX = B$ has infinitely many solutions, else no solution.

Note : A system of equation is called consistent if it has atleast one solution. If the system has no solution, then it is called inconsistent.

SYSTEM OF LINEAR HOMOGENEOUS SIMULTANEOUS EQUATIONS

Consider the system of linear homogeneous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$.

In this case, system of equations is always consistent as $x = y = z = 0$ is always a solution. If the system has unique solution (the case when coefficient determinant $\neq 0$), then $x = y = z = 0$ is the only solution (called trivial solution). However if the system has coefficient determinant = 0, then the system has infinitely many solutions. Hence in this case we get solutions other than trivial solution also and we say that we have non-trivial solutions.

PROBLEMS

Single Correct Answer Type

1. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 1$

(a) $a^2 + b^2 + c^2$ (b) $(a + b)(b + c)(c + a)$
(c) $(a - b)(b - c)(c - a)$ (d) None of these

2. The determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$ is not equal to

(a) $\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 3 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ 1 & 9 & 6 \end{vmatrix}$ (d) $\begin{vmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 10 & 3 & 6 \end{vmatrix}$

3. $\begin{vmatrix} 1+i & 1-i & i \\ 1-i & i & 1+i \\ i & 1+i & 1-i \end{vmatrix} =$

(a) $-4-7i$ (b) $4+7i$ (c) $3+7i$ (d) $7+4i$

4. $\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix} =$

(a) $a^2+b^2+c^2-3abc$ (b) $3ab$
(c) $3a+5b$ (d) 0

5. The roots of the equation $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = 0$ are

(a) 0, – 3 (b) 0, 0, – 3
(c) 0, 0, 0, – 3 (d) None of these

6. $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} =$

(a) 2 (b) – 2
(c) x^2-2 (d) None of these

7. $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix} =$

(a) abc (b) $1/abc$
(c) $ab+bc+ca$ (d) 0

8. If ω is a cube root of unity, then

$\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} =$

(a) x^3+1 (b) $x^3+\omega$ (c) $x^3+\omega^2$ (d) x^3

9. If a, b and c are non-zero numbers, then

$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ is equal to

(a) abc (b) $a^2b^2c^2$
(c) $ab+bc+ca$ (d) 0

10. If a, b, c are positive integers, then the determinant

$\Delta = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$ is divisible by

(a) x^3 (b) x^2
(c) $(a^2+b^2+c^2)$ (d) None of these

11. $2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ac & c^2-ab \end{vmatrix} =$

(a) 0 (b) 1 (c) 2 (d) $3abc$

12. If $D_p = \begin{vmatrix} p & 15 & 8 \\ p^2 & 35 & 9 \\ p^3 & 25 & 10 \end{vmatrix}$, then $D_1+D_2+D_3+D_4+D_5 =$

(a) 0 (b) 25
(c) 625 (d) None of these

13. If a, b, c are different and $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$, then

(a) $a+b+c=0$ (b) $abc=1$
(c) $a+b+c=1$ (d) $ab+bc+ca=0$

14. If $ab+bc+ca=0$ and $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$,

then one of the values of x is

(a) $(a^2+b^2+c^2)^{\frac{1}{2}}$ (b) $\left[\frac{3}{2}(a^2+b^2+c^2)\right]^{\frac{1}{2}}$

(c) $\left[\frac{1}{2}(a^2+b^2+c^2)\right]^{\frac{1}{2}}$ (d) None of these

15. If $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = kabc(a+b+c)^3$

then the value of k is

(a) – 1 (b) 1 (c) 2 (d) –2

16. The number of solutions of the equations $x+4y-z=0$, $3x-4y-z=0$, $x-3y+z=0$ is

(a) 0 (b) 1 (c) 2 (d) Infinite

17. If $\Delta(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then the value of $\frac{d^n}{dx^n}[\Delta(x)]$ at $x = 0$ is

(a) – 1 (b) 0
(c) 1 (d) Dependent of a

18. The value of a for which the system of equations $a^3x + (a + 1)^3y + (a + 2)^3z = 0$, $ax + (a + 1)y + (a + 2)z = 0$, $x + y + z = 0$, has a non zero solution is

(a) – 1 (b) 0
(c) 1 (d) None of these

19. $x_1 + 2x_2 + 3x_3 = 2ax_1 + 3x_2 + x_3 = 3bx_1 + x_2 + 2x_3 = c$ this system of equations has

(a) Infinite solution (b) No solution
(c) Unique solution (d) None of these

20. The value of $\sum_{n=1}^{N} U_n$, if $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}$ is

(a) 0 (b) 1
(c) –1 (d) None of these

21. If $D_r = \begin{vmatrix} 2^{r-1} & 2\cdot 3^{r-1} & 4\cdot 5^{r-1} \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, then the value of $\sum_{r=1}^{n} D_r =$

(a) 1 (b) –1
(c) 0 (d) None of these

22. The number of values of k for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k – 1$ has infinitely many solutions, is

(a) 0 (b) 1 (c) 2 (d) Infinite

23. If $a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative, then $\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$ is

(a) Positive (b) $(ac – b^2)(ax^2 + 2bx + c)$
(c) Negative (d) 0

24. For what value of λ, the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = 12$ is inconsistent

(a) $\lambda = 1$ (b) $\lambda = 2$ (c) $\lambda = -2$ (d) $\lambda = 3$

25. If x is a positive integer, then

$\Delta = \begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix}$ is equal to

(a) $2(x!)(x + 1)!$ (b) $2(x!)(x + 1)!(x + 2)!$
(c) $2(x!)(x + 3)!$ (d) None of these

26. The value of λ for which the system of equations $2x – y – z = 12$, $x – 2y + z = –4$, $x + y + \lambda z = 4$ has no solution is

(a) 3 (b) –3 (c) 2 (d) –2

27. If a, b, c are respectively the p^{th}, q^{th} r^{th} terms of an A.P., then $\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix} =$

(a) 1 (b) –1 (c) 0 (d) pqr

28. If the system of linear equation $x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$ has a non zero solution, then a, b, c

(a) are in A.P. (b) are in G.P.
(c) are in H.P. (d) satisfy $a + 2b + 3c = 0$

29. If a system of the equation $(\alpha + 1)^3x + (\alpha + 2)^3y – (\alpha + 3)^3 = 0$, $(\alpha + 1)x + (\alpha + 2)y – (\alpha + 3) = 0$ and $x + y – 1 = 0$ is consistent. What is the value of α?

(a) 1 (b) 0 (c) –3 (d) –2

30. If the matrix $\begin{bmatrix} 1 & 3 & \lambda+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$ is singular, then $\lambda =$

(a) –2 (b) 4 (c) 2 (d) –4

31. A, B are n-rowed square matrices such that $AB = O$ and B is non-singular. Then

(a) $A \neq O$ (b) $A = O$
(c) $A = I$ (d) None of these

32. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2 =$

(a) $2AB$ (b) $2BA$ (c) $A + B$ (d) AB

33. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A^4 =$

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

34. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, then $A^2=$

(a) $\begin{bmatrix}8 & -5\\ -5 & 3\end{bmatrix}$ (b) $\begin{bmatrix}8 & -5\\ 5 & 3\end{bmatrix}$

(c) $\begin{bmatrix}8 & -5\\ -5 & -3\end{bmatrix}$ (d) $\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$

35. If $A=\begin{bmatrix}1 & 3 & 0\\ -1 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}, B=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ -1 & 1 & 2\end{bmatrix}$, then $AB=$

(a) $\begin{bmatrix}5 & 9 & 13\\ -1 & 2 & 4\\ -1 & -2 & 4\end{bmatrix}$ (b) $\begin{bmatrix}5 & 9 & 13\\ -1 & 2 & 4\\ -2 & 2 & 4\end{bmatrix}$

(c) $\begin{bmatrix}1 & 2 & 4\\ -1 & 2 & 4\\ -2 & 2 & 4\end{bmatrix}$ (d) None of these

36. If $A=\begin{bmatrix}1 & 2\\ -3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0\\ 2 & 3\end{bmatrix}$, then

(a) $A^2=A$ (b) $B^2=B$
(c) $AB\neq BA$ (d) $AB=BA$

37. In order that the matrix $\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 3 & \lambda & 5\end{bmatrix}$ be non-singular, λ should not be equal to

(a) 1 (b) 2 (c) 3 (d) 4

38. If $A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$, then the value of A^{40} is

(a) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$

MULTIPLE CORRECT ANSWER TYPE

39. Let A_n is a $n\times n$ matrix in which diagonal elements are 1, 2, 3, ..., n (*i.e.*, $a_{11}=1, a_{22}=2, a_{33}=3, \ldots a_{ii}=i, \ldots. a_{nn}=n$) and all other elements are equal to n, then

(a) A_n is singular for all 'n'
(b) A_n is non-singular for all 'n'
(c) det. $A_5=120$
(d) det. $A_n=0$

40. Let $a_1, a_2, a_3, \ldots$ be real numbers which are in arithmetic progression with common difference $d\neq 0$. Then

(a) $A=\begin{bmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_5 & a_6 & a_7\end{bmatrix}$ is singular

(b) $B=\begin{bmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_2 & a_4 & a_6\end{bmatrix}$ is non-singular

(c) The system of equations $a_1x+a_2y+a_3z=0$ $a_3x+a_1y+a_2z=0$, $a_4x+a_5y+a_6z=0$ has unique solution

(d) The system of equations $a_1x+a_2y+a_3z=0$, $a_4x+a_5y+a_6z=0$, $a_7x+a_8y+a_9z=0$ has infinitely many solutions

41. If $A=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$ and $A^{2012}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then which of the following is/are correct?

(a) $a=d$ (b) $a+b+c+d=4026$
(c) $a^2+b^2+d^2=2$ (d) $b=2012$

42. Which of the following statements are FALSE?

(a) If A and B are square matrices of the same order such that $ABAB=0$, it follows that $BABA=0$.
(b) Let A and B be different $n\times n$ matrices with real numbers. If $A^3=B^3$ and $A^2B=B^2A$, then A^2+B^2 is invertible.
(c) If A is a square, non-singular and symmetric matrix, then $\left((A^{-1})^{-1}\right)^{-1}$ is skew symmetric.
(d) The matrix of the product of two invertible square matrices of the same order is also invertible.

43. The system of equation is $x-y\cos\theta+z\cos 2\theta=0$, $x\cos 2\theta-y+z\cos\theta=0$, $x\cos 2\theta-y\cos\theta+z=0$ has non-trivial solution for θ equals to

(a) $\dfrac{8\pi}{3}$ (b) $\dfrac{\pi}{6}$ (c) $\dfrac{2\pi}{3}$ (d) $\dfrac{\pi}{12}$

44. *If* $A=\begin{bmatrix}a & b & c\\ x & y & z\\ p & q & r\end{bmatrix}$ and $B=\begin{bmatrix}q & -b & y\\ -p & a & -x\\ r & -c & z\end{bmatrix}$, then

(a) $|A|=|B|$
(b) $|A|=-|B|$
(c) $|A|=2|B|$
(d) A is invertible if and only if B is invertible

COMPREHENSION TYPE

Paragraph for Question No. 45 to 47

A and B are two matrices of same order 3×3, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 8 \end{pmatrix} \text{and } B = \begin{pmatrix} 3 & 2 & 5 \\ 2 & 3 & 8 \\ 7 & 2 & 9 \end{pmatrix}$$

45. The value of $|\text{adj}(\text{adj } A)|$ is equal to
(a) 9 (b) 25
(c) 81 (d) None of these

46. The value of $|\text{adj}(AB)|$ is equal to
(a) 24 (b) 24^2 (c) 24^3 (d) 65

47. The value of $|(\text{adj}(\text{adj}(\text{adj}(\text{adj}A))))|$ is equal to
(a) 2^4 (b) 2^9
(c) 2^{19} (d) None of these

Paragraph for Question No. 48 to 50

Two $n \times n$ square matrices A and B are said to be similar if their exists a non-singular matrix P such that $P^{-1}BP = A$

48. If A and B are similar matrices such that

$$|A| = |\text{adj}(\text{adj}(Q))| \text{ where } Q = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}, \text{ then}$$

(a) $|B| = 10^{-4}$ (b) $|A| + |B| = 2 \times 10^4$
(c) $|B| = 10^3$ (d) None of these

49. If A and B are similar and B and C are similar, then
(a) AB and BC are similar (b) A and C are similar
(c) $A + C$ and B are similar (d) None of these

50. If A and B are two non-singular matrices, then
(a) A is similar to B (b) AB is similar to BA
(c) AB is similar to $A^{-1}B$ (d) None of these

Paragraph for Question No. 51 to 52

A square matrix A such that $A^\theta = A$ is called Hermitian matrix *i.e.* $a_{ij} = \bar{a}_{ji}$ for all values of i and j and a square matrix A such that $A^\theta = -A$ is called Skew–hermitian matrix *i.e.* $a_{ij} = -\bar{a}_{ji}$ for all values of i and j where A^θ is conjugate transpose matrix.

Let $f : M \to \{1, -1\}$, M is set of all hermitian or skew–hermitian matrixes, be a function defined as

$$f(A) = \begin{cases} 1 & A \text{ is hermitian} \\ -1 & A \text{ is skew} - \text{hermitian} \end{cases}$$

51. $f(A - A^\theta) =$
(a) +1 (b) –1
(c) +1 when $f(A) > 0$ (d) –1 when $f(A) > 0$

52. Let $Y = A^n$
(a) if $f(Y) = 1$ then $f(A) = 1$
(b) if $f(Y) = -1$ then $f(A) = -1$
(c) $f(Y) \cdot f(A) = -1$ if n is odd
(d) $f(Y) \cdot f(A) = 1$ if n is even

53. Let $A = [a_{ij}]_{4\times4}$ be a matrix such that $\arg(a_{ij}) \in \left[0, \frac{\pi}{2}\right]$ and $f(A) = -1$, and if $[b_{ij}]_{2\times2}$ be a matrix defined by $b_{ij} = a_{ii} + a_{jj}$ $\forall i, j$ then
(a) B is a unit matrix (b) Trace $(B) = 0$
(c) B is a null matrix (d) Det $(B) \geq 0$

MATRIX – MATCH TYPE

54. Match the following :

	Column I		Column II
(A)	A is a real skew symmetric matrix such that $A^2 + I = 0$. Then	(p)	$BA - AB$
(B)	A is a matrix such that $A^2 = A$. If $(I + A)^n = I + \lambda A$, then λ equals $(n \in N)$	(q)	A is of even order
(C)	If for a matrix A, $A^2 = A$, and $B = I - A$, then $AB + BA + I - (I - A)^2$ equals	(r)	A
(D)	A is a matrix with complex entries and A^* stands for transpose of complex conjugate of A. If $A^* = A$ and $B^* = B$, then $(AB - BA)^*$ equals	(s)	$2^n - 1$
		(t)	${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$

55. Match the following :

	Column I		Column II
(A)	Let $\lvert A\rvert = \lvert a_{ij}\rvert_{3\times3} \neq 0$. Each element a_{ij} is multiplied by k^{i-j}. Let $\lvert B\rvert$ the resulting determinant, where $k_1\lvert A\rvert + k_2\lvert B\rvert = 0$. Then $k_1 + k_2 =$	(p)	0
(B)	The maximum value of a third order determinant each of its entries are ±1 equals	(q)	4
(C)	$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$	(r)	1

(D)	$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$ $= Ax + B$ then $A + 2B$ is	(s)	$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

INTEGER ANSWER TYPE

56. If the integers a, b, c in order are in A.P., lying between 1 and 9 and $a23$, $b53$, and $c83$ are three-digit numbers, then the value of the determinant $\begin{vmatrix} 2 & 5 & 8 \\ a23 & b53 & c83 \\ a & b & c \end{vmatrix}$ is

57. If the adjoint of a 3 × 3 matrix P is $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 4 & 7 & 3 \end{bmatrix}$, then the sum of squares of possible values of determinant of P is

58. If $A=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $\det(A^{2005})$ equals to

59. Let $f(x)=\begin{vmatrix} x & 1 & 1 \\ \sin 2\pi x & 2x^2 & 1 \\ x^3 & 3x^4 & 1 \end{vmatrix}$. If $f(x)$ be an odd function and its odd values is equal to $g(x)$, then find the value of λ given that $\lambda f(1)g(1) = 4$

60. If $A=\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $A^2 = 8A + kI_2$, then find the value of $|k|$.

SOLUTIONS

1. (c) : $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$

(By $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$)

$=(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$

$=(a-b)\,(b-c)\begin{vmatrix} 0 & 0 & a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$, (By $R_1 \to R_1 - R_2$)

$=(a-b)(b-c)(a-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$

$= (a - b)(b - c)(a - c)\cdot(-1) = (a - b)(b - c)(c - a)$

2. (d) : $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \end{vmatrix}$ by $C_1 \to C_1 + C_2$

$= \begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ 1 & 9 & 6 \end{vmatrix}$, by $C_2 \to C_2 + C_3$

$= \begin{vmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 10 & 3 & 6 \end{vmatrix}$, by $C_1 \to C_1 + C_2 + C_3$

3. (b) : $\Delta = (2+i)\begin{vmatrix} 1 & 1 & i \\ 1 & 1+2i & 1+i \\ 1 & 2 & 1-i \end{vmatrix}$

$= (2+i)\begin{vmatrix} 0 & -2i & -1 \\ 0 & -1+2i & 2i \\ 1 & 2 & 1-i \end{vmatrix}$

(By $R_1 \to R_1 + R_2$, $R_2 \to R_2 + R_3$)

$= (2 + i)\{-4i^2 + (-1 + 2i)\} = (2 + i)(4 - 1 + 2i)$

$= (2 + i)(3 + 2i) = 4 + 7i$.

4. (d) : We have, $\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix}$

$= \begin{vmatrix} a+b & a+2b & a+3b \\ b & b & b \\ 2b & 2b & 2b \end{vmatrix} = 0 \quad \left\{\text{by } \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix}\right\}$

5. (b) : We have, $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3+x & 0 & 1 \\ 3+x & x & 1 \\ 3+x & -x & 1+x \end{vmatrix} = 0, \begin{pmatrix} C_1 \to C_1 + C_2 + C_3 \\ C_2 \to C_2 - C_3 \end{pmatrix}$

$\Rightarrow (x+3)\begin{vmatrix} 1 & 0 & 1 \\ 1 & x & 1 \\ 1 & -x & 1+x \end{vmatrix} = 0$

$\Rightarrow (x+3)\begin{vmatrix} 1 & 0 & 1 \\ 0 & x & 0 \\ 0 & -x & x \end{vmatrix} = 0, \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$\Rightarrow (x + 3)x^2 = 0 \Rightarrow x = 0, 0, -3.$

6. (b): $\Delta = \begin{vmatrix} -1 & -2 & x+4 \\ -2 & -3 & x+8 \\ -3 & -4 & x+14 \end{vmatrix} \quad \begin{pmatrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{pmatrix}$

$$= \begin{bmatrix} -1 & -1 & x \\ -2 & -1 & x \\ -3 & -1 & x+2 \end{bmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 + 4C_1 \end{pmatrix}$$

$= -(-x - 2 + x) + 1.\ (-2x - 4 + 3x) + x(2 - 3)$

$= 2 + x - 4 - x = -2$

7. (d)

8. (d): $\Delta = \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$

$$= \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ x+1+\omega+\omega^2 & x+\omega^2 & 1 \\ x+1+\omega+\omega^2 & 1 & x+\omega \end{vmatrix} (C_1 \to C_1 + C_2 + C_3)$$

$$= x\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x+\omega^2 & 1 \\ 1 & 1 & x+\omega \end{vmatrix} \quad (\because\ 1+\omega+\omega^2 = 0)$$

$= x[1\{(x + \omega^2)(x + \omega)-1\} + \omega\{1- (x + \omega)\}$
$+ \omega^2\{1 - (x + \omega^2)\}]$

$= x(x^2 + \omega x + \omega^2 x + \omega^3 - 1 + \omega - \omega x - \omega^2 + \omega^2$
$- \omega^2 x - \omega^4) = x^3 \quad (\because\ \omega^3 = 1).$

9. (d): Multiplying R_1 by a, R_2 by b and R_3 by c, we have $\Delta = \dfrac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$

$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ac & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix} = abc\begin{vmatrix} bc & 1 & \Sigma ab \\ ac & 1 & \Sigma ab \\ ab & 1 & \Sigma ab \end{vmatrix}$$

(by $C_3 \to C_3 + C_1$)

$$= abc.\Sigma ab\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} = 0, \text{ [Since } C_2 \sim C_3]$$

10. (b): $\Delta = \dfrac{1}{abc}\begin{vmatrix} a^3+ax & a^2b & a^2c \\ ab^2 & b^3+bx & b^2c \\ c^2a & c^2b & c^3+cx \end{vmatrix}$

$$= \begin{vmatrix} a^2+x & a^2 & a^2 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

$$= (a^2+b^2+c^2+x)\times\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

(Applying $R_1 \to R_1 + R_2 + R_3$)

$$= (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & x & 0 \\ c^2 & 0 & x \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$= x^2(a^2 + b^2 + c^2 + x)$

Hence Δ is divisible by x^2 as well as by x.

11. (a): We have $2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ac & c^2-ab \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} - \frac{2}{abc}\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ abc & abc & abc \end{vmatrix}$$

Applying $C_1(a)$, $C_2(b)$, $C_3(c)$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} - \frac{2}{abc}(abc)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

12. (d)

13. (b): We have, $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}-\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}=0$$

$$\Rightarrow (abc-1)\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}=0$$

Since a, b, c are different, so $\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}\neq 0$

Hence $abc - 1 = 0$ *i.e.*, $abc = 1$

14. (a) : Applying $C_1 \to C_1 + C_2 + C_3$

$$\therefore (a+b+c-x)\begin{vmatrix}1 & c & b\\1 & b-x & a\\1 & a & c-x\end{vmatrix}=0$$

$\Rightarrow (a + b + c - x)[\{(b - x)(c - x) - a^2\}$
$+ c(a - c + x)\} + b(a - b + x)\}] = 0$

$\Rightarrow (a + b + c - x)[bc - cx - bx + x^2 - a^2 + ca - c^2$
$+ cx + ab - b^2 + bx] = 0$

$\Rightarrow (a + b + c - x)[x^2 - (a^2 + b^2 + c^2) + ab + bc + ca] = 0$

$(a + b + c - x)[x^2 - (a^2 + b^2 + c^2)] = 0$

$\therefore x = a + b + c$ and $(a^2 + b^2 + c^2)^{1/2}$

15. (c) : Operate $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$ and take out $a + b + c$ from C_2 as well as from C_3 to get

$$\Delta=(a+b+c)^2\begin{vmatrix}(b+c)^2 & a-b-c & a-b-c\\b^2 & c+a-b & 0\\c^2 & 0 & a+b-c\end{vmatrix}$$

Operate $R_1 \to R_1 - R_2 - R_3$

$$=(a+b+c)^2\begin{vmatrix}2bc & -2c & -2b\\b^2 & c+a-b & 0\\c^2 & 0 & a+b-c\end{vmatrix}$$

$\left(\text{Operate } C_2 \to C_2+\frac{1}{b}C_1 \text{ and } C_3 \to C_3+\frac{1}{c}C_1\right)$

$$=(a+b+c)^2\begin{vmatrix}2bc & 0 & 0\\b^2 & c+a & \frac{b^2}{c}\\c^2 & \frac{c^2}{b} & a+b\end{vmatrix}$$

$= (a + b + c)^2[2bc\{(a + b)(c + a) - bc\}]$
$= 2abc(a + b + c)^3$.

16. (b) : The given system of homogeneous equations

has $\Delta=\begin{vmatrix}1 & 4 & -1\\3 & -4 & -1\\1 & -3 & 1\end{vmatrix}=1(-4-3)-4(3+1)-1(-9+4)$

$= -7 - 16 + 5 \neq 0$

There exists only one trivial solution.

17. (b) : $$\frac{d^n}{dx^n}[\Delta(x)]=\begin{vmatrix}\frac{d^n}{dx^n}x^n & \frac{d^n}{dx^n}\sin x & \frac{d^n}{dx^n}\cos x\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ a & a^2 & a^3\end{vmatrix}$$

$$=\begin{vmatrix}n! & \sin\left(x+\frac{n\pi}{2}\right) & \cos\left(x+\frac{n\pi}{2}\right)\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ a & a^2 & a^3\end{vmatrix}$$

$$=[\Delta^n(x)]_{x=0}=\begin{vmatrix}n! & \sin\left(0+\frac{n\pi}{2}\right) & \cos\left(0+\frac{n\pi}{2}\right)\\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)\\ a & a^2 & a^3\end{vmatrix}=0$$

{Since $R_1 \equiv R_2$}

18. (a) : The system will have a non-zero solution, if

$$\Delta\equiv\begin{vmatrix}a^3 & (a+1)^3 & (a+2)^3\\a & a+1 & a+2\\1 & 1 & 1\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}a^3 & 3a^2+3a+1 & 3(a+1)^2+3(a+1)+1\\a^2 & 1 & 1\\1 & 0 & 0\end{vmatrix}=0$$

(by $C_2 \to C_2 - C_1, C_3 \to C_3 - C_2$)

$\Rightarrow 3a^2 + 3a + 1 - \{3(a + 1)^2 + 3(a + 1)+1\}$

(expanding along R_3)

$\Rightarrow -6(a + 1) = 0 \Rightarrow a = -1$

19. (c) : We have, $x_1 + 2x_2 + 3x_3 = c$

$2ax_1 + 3x_2 + x_3 = c$, $3bx_1 + x_2 + 2x_3 = c$

Let $a = b = c = 1$

Then $D=\begin{vmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{vmatrix}=1(5)-2(1)+3(-7)=-18\neq 0$

$$D_x = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -3$$

Similarly $D_y = D_z = -3$. Now, $x = \frac{D_x}{D} = \frac{1}{6}. y = z = \frac{1}{6}$

Hence $D \neq 0$, $x = y = z$, *i.e.*, unique solution

20. (a) : $\displaystyle\sum_{n=1}^{N} U_n = \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left\{\frac{N(N+1)}{2}\right\}^2 & 3N^2 & 3N \end{vmatrix}$

$$= \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 5 \\ 4N+2 & 2N+1 & 2N+1 \\ 3N(N+1) & 3N^2 & 3N \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 1 & 6 \\ 4N+2 & 2N+1 & 4N+2 \\ 3N(N+1) & 3N^2 & 3N(N+1) \end{vmatrix} = 0$$

{Applying $C_3 \to C_3 + C_2$}

21. (c) : $D_r = \begin{vmatrix} 2^{r-1} & 2\cdot 3^{r-1} & 4\cdot 5^{r-1} \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$

$$\Rightarrow \sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & \sum_{r=1}^{n} 2\cdot 3^{r-1} & \sum_{r=1}^{n} 4\cdot 5^{r-1} \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

$$\Rightarrow \sum_{r=1}^{n} D_r = \begin{vmatrix} 2^n - 1 & 3^n - 1 & 5^n - 1 \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

Since we know that $\displaystyle\sum_{r=1}^{n} 2^{r-1} = \frac{2^n - 1}{2-1} = 2^n - 1$

$\Rightarrow \displaystyle\sum_{r=1}^{n} D_r = 0 \quad (\because R_1 \sim R_3)$

22. (b) : For infinitely many solutions, the two equations must be identical

$\Rightarrow \frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1}$

$\Rightarrow$ $(k + 1)(k + 3) = 8k$ and $8(3k - 1) = 4k(k + 3)$

$\Rightarrow$ $k^2 - 4k + 3 = 0$ and $k^2 - 3k + 2 = 0$

By cross multiplication, $\frac{k^2}{-8+9} = \frac{k}{3-2} = \frac{1}{-3+4}$

$k^2 = 1$ and $k = 1$; $\therefore$ $k = 1$.

23. (c) : Let $\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$

Applying $R_3 \to R_3 - xR_1 - R_2$; we get

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ 0 & 0 & -(ax^2+2bx+c) \end{vmatrix}$$

Now, $b^2 - ac < 0$ and $a > 0$

$\Rightarrow$ Discriminant of $ax^2 + 2bx + c$ is –ve and $a > 0$

$\Rightarrow$ $(ax^2 + 2bx + c) > 0$ for all $x \in R$

$\Rightarrow$ $\Delta = (b^2 - ac)(ax^2 + 2bx + c) < 0$, *i.e.*–ve.

24. (d) : Given system will be inconsistent when $D = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ -1 & -1 & 3 \\ -1 & 2-\lambda & \lambda \end{vmatrix} = 0 \Rightarrow -1(2-\lambda) - 1 = 0 \Rightarrow \lambda = 3$$

25. (b) : $\Delta = \begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix}$

$$= x!(x+1)!(x+2)! \begin{vmatrix} 1 & (x+1) & (x+2)(x+1) \\ 1 & (x+2) & (x+3)(x+2) \\ 1 & (x+3) & (x+4)(x+3) \end{vmatrix}$$

Applying $R_1 \to R_2 - R_1$, $R_2 \to R_3 - R_2$ we get

$$= x!(x+1)!(x+2)! \begin{vmatrix} 0 & 1 & 2(x+2) \\ 0 & 1 & 2(x+3) \\ 1 & (x+3) & (x+4)(x+3) \end{vmatrix}$$

$= 2x!(x + 1)!(x + 2)!$,(Expanding along C_1).

26. (d) : The coefficient determinant

$$D = \begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = -3\lambda - 6$$

For no solution, the necessary condition is $D = 0$

i.e., $-3\lambda - 6 = 0 \Rightarrow \lambda = -2$

27. (c): Let first term $= A$ and common difference $= D$

$\therefore\quad a = A + (p-1)D,\ b = A + (q-1)D,\ c = A + (r-1)D$

$$\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix} = \begin{vmatrix} A+(p-1)D & p & 1 \\ A+(q-1)D & q & 1 \\ A+(r-1)D & r & 1 \end{vmatrix}$$

Operate $C_1 \to C_1 - DC_2 + DC_3$

$$= \begin{vmatrix} A & p & 1 \\ A & q & 1 \\ A & r & 1 \end{vmatrix} = A\begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} = 0$$

28. (c) : $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$, [Applying $C_2 \to C_2 - 2C_3$]

$$\Rightarrow \begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix} = 0,\ [R_3 \to R_3 - R_2, R_2 \to R_2 - R_1]$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & a \\ 0 & b & b-a \\ 0 & 2c-b & c-b \end{vmatrix} = 0; \Rightarrow b(c-b) - (b-a)(2c-b) = 0$$

On simplification, $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$, H.P.

29. (d) : For consistent solution, $|A| = 0$

i.e., $\begin{vmatrix} (\alpha+1)^3 & (\alpha+2)^3 & -(\alpha+3)^3 \\ \alpha+1 & \alpha+2 & -(\alpha+3) \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow\quad 6\alpha + 12 = 0 \ \Rightarrow\ \alpha = -2$.

30. (b)

31. (b) : Since $|B| \neq 0 \Rightarrow B^{-1}$ exists $\therefore\ AB = 0$

$\Rightarrow\quad (AB)B^{-1} = OB^{-1} \ \Rightarrow\ A(BB^{-1}) = 0$

$\Rightarrow\quad AI = O \ \Rightarrow\ A = O$.

32. (c) : We have $AB = B$ and $BA = A$

Therefore $A^2 + B^2 = AA + BB = A(BA) + B(AB)$

$= (AB)A + (BA)B = BA + AB = A + B$

$(\because AB = B \text{ and } BA = A)$

33. (a) : We have $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$\therefore\quad A^4 = A^2 \cdot A^2 = I_2 \cdot I_2 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

34. (d) : We have $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$.

35. (b) :

36. (c) : Since $A^2 = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -3 & -6 \end{bmatrix} \neq A$

$B^2 = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 9 \end{bmatrix} \neq B$

Now $AB = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 0 \end{bmatrix}$

and $BA = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -7 & 4 \end{bmatrix}$

Obviously, $AB \neq BA$.

37. (d) : Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & \lambda & 5 \end{bmatrix}$ be non singular

only if $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & \lambda & 5 \end{vmatrix} \neq 0$

$\Rightarrow\quad 1(25 - 6\lambda) - 2(20 - 18) + 3(4\lambda - 15) \neq 0$

$\Rightarrow\quad 25 - 6\lambda - 4 + 12\lambda - 45 \neq 0$

$\Rightarrow\quad 6\lambda - 24 \Rightarrow 0 \Rightarrow \lambda \neq 4.$

38. (b) : $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\Rightarrow (A^2)^{20} = A^{40} = (I)^{20} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

39. (b, c) : $A_n = \begin{pmatrix} 1 & n & n\ldots & n \\ n & 2 & n\ldots & n \\ n & n & 3\ldots & n \\ n & n & n\ldots & n \end{pmatrix}$

A_n is non-singular for all n'

$|A_n| = (-1)^{n+1} n!$

40. (a, c, d) : Determinants in (a), (b) and (d) are zero, and in (c) the determinant is non zero.

41. (a, b, c) : $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

Hence, $A^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}$

42. (a, b, c) : (a) $A=\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}; B=\begin{bmatrix}0&0&1\\1&0&0\\0&0&0\end{bmatrix}$

is a counter example. Therefore, (a) is wrong.

(b) We have $(A^2+B^2)(A-B)=A^3-B^3-A^2B+B^2A=0$ and $A-B\neq 0 \Rightarrow A^2+B^2$ is not invertible. Therefore (b) is also wrong.

(c) We know that $A^T=A$

$$\left((A^{-1})^{-1}\right)^{-1}=A^{-1}$$

$(A^{-1})^T=(A^T)^{-1}=A^{-1}$

So, $\left((A^{-1})^{-1}\right)^{-1}$ is also symmetric. Therefore (c) is also wrong.

(d) $|A|\neq 0$, $|B|\neq 0 \Rightarrow AB\neq 0$

So, AB is invertible. Therefore (d) is correct.

43. (a, c) : $\begin{vmatrix}1&-\cos\theta&\cos 2\theta\\\cos 2\theta&-1&\cos\theta\\\cos 2\theta&-\cos\theta&1\end{vmatrix}=0 \Rightarrow \theta=\dfrac{2\pi}{3}$

44. (b, d) : $|B|=\begin{vmatrix}q&-b&y\\-p&a&-x\\r&-c&z\end{vmatrix}$

$$=-\begin{vmatrix}q&b&y\\-p&-a&-x\\r&c&z\end{vmatrix}\text{ (operate } C_2)$$

$$=\begin{vmatrix}q&b&y\\p&a&x\\r&c&z\end{vmatrix}=-\begin{vmatrix}p&a&x\\q&b&y\\r&c&z\end{vmatrix}\text{ (on } R_2)\text{ (on } R_1\leftrightarrow R_2)$$

$\Rightarrow |B|=-|A|$

45. (d) : $|\operatorname{adj}(\operatorname{adj}A)|=|A|^{(n-1)^2}=|A|^4=(-1)^4=1$

46. (b) : $AB=\begin{pmatrix}1&2&3\\2&3&4\\5&6&8\end{pmatrix}\begin{pmatrix}3&2&5\\2&3&8\\7&2&9\end{pmatrix}$

$$=\begin{pmatrix}28&14&48\\40&21&70\\83&44&145\end{pmatrix}\quad\therefore\ |AB|=\begin{vmatrix}28&14&48\\40&21&70\\83&44&145\end{vmatrix}$$

Applying $C_1\to C_1-2C_2$, $C_3\to C_3-3C_2$, we get

$$|AB|=\begin{vmatrix}0&14&6\\-2&21&7\\-5&44&13\end{vmatrix}=\begin{vmatrix}0&-4&6\\-2&0&7\\-5&5&13\end{vmatrix}(C_2\to C_2-3C_3)$$

$=0+2(-52-30)-5(-28)=-164+140=-24$

$|\text{adj}AB|=|AB|^{n-1}=|AB|^2=(24)^2$

47. (d)

48. (b) : $|Q|=10$

$$|\operatorname{adj}(\operatorname{adj}Q)|=|Q|^{(3-1)^2}=10^4=|A|$$

$P^{-1}BP=A \Rightarrow |B|=|A| \Rightarrow |A|+|B|=2\times 10^4$

49. (b) : $P^{-1}BP=A$ ($\because$ A and B are similar)

$R^{-1}CR=B$ ($\because$ B and C are similar)

$\Rightarrow P^{-1}(R^{-1}CR)P=A \Rightarrow (P^{-1}R^{-1})C(RP)=A$

$\Rightarrow (RP)^{-1}C(RP)=A \Rightarrow A$ and C are similar

50. (b) : $A^{-1}(AB)A=BA$

AB and BA are similar

51. (b) : $X=A-A^{\theta}$ is a skew hermitian

($\because$ $X^{\theta}=(A-A^{\theta})^{\theta}=A^{\theta}-A=-X$

$f(x)=-1$

52. (b) : $Y^{\theta}=(A^n)^{\theta}=(A^{\theta})^n$

$$=\begin{cases}A^n=Y & \text{if } A \text{ is hermitian}\\(-A)^n=\begin{cases}Y\text{ if } A \text{ is skew hermitian and } n \text{ is even}\\-Y\text{ if } A \text{ is skew hermitian and } n \text{ is odd}\end{cases}\end{cases}$$

53. (d) : A is skew hermitian $\Rightarrow$ diagonal elements are purely imaginary or zero

Also $\arg a_{ij}\in\left[0,\dfrac{\pi}{2}\right] \Rightarrow \operatorname{Im} a_{ij}\geq 0$

$a_{ii}=iy_{ii} \Rightarrow a_{11}=iy_{11}$, $a_{22}=iy_{22}$, $a_{33}=iy_{33}$, $a_{44}=iy_{44}$
(where $y_{11}, y_{22}, y_{33}, y_{44}\geq 0$)

$$B=\begin{bmatrix}2iy_{11}&iy_{11}+iy_{22}\\iy_{11}+iy_{22}&2iy_{22}\end{bmatrix}\Rightarrow |B|=(y_{11}-y_{22})^2\geq 0$$

54. A - q; B - s,t; C - r; D - p

(A) $A^2=-I$. $|A^2|=-1$. Thus, A is of even order.

(B) $(I+A)^n=C_0I+C_1IA+C_2IA^2+\ldots+C_nIA^n$
$\quad=C_0I+C_1A+C_2A+\ldots+C_nA$

$\therefore\ \lambda=2^n-1$

(C) $A^2=A$ and $B=I-A$

$AB+BA+I-(I+A^2-2A)$
$\quad=AB+BA-A+2A=AB+BA+A$
$\quad=A(I-A)+(I-A)A+A$
$\quad=A-A+A-A+A=A$

(D) $A^*=A$, $B^*=B$

$(AB-BA)^*=B^*A^*-A^*B^*=BA-AB$

55. A - p; B - q; C - r; D - p

(A) $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|B| = \begin{vmatrix} a_{11} & k^{-1}a_{12} & k^{-2}a_{13} \\ ka_{21} & a_{22} & k^{-1}a_{23} \\ k^2a_{31} & ka_{32} & a_{33} \end{vmatrix}$$

$$= \frac{1}{k^3}\begin{vmatrix} k^2a_{11} & ka_{12} & a_{13} \\ k^2a_{21} & ka_{22} & a_{23} \\ k^2a_{31} & ka_{32} & a_{33} \end{vmatrix} = |A|$$

$k_1|A| + k_2|B| = 0$; $k_1 + k_2 = 0$

(B) $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$

(C) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$

$$\Rightarrow \sin^2\gamma - \cos\alpha(\cos\alpha - \cos\beta\cos\gamma) + \cos\beta(\cos\alpha\cos\gamma - \cos\beta)$$

$$= -\cos\alpha(-\cos\beta\cos\gamma) + \cos\beta(\cos\alpha\cos\gamma)$$

$$\Rightarrow \sin^2\gamma - \cos^2\alpha + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\beta = 2\cos\alpha\cos\beta\cos\gamma$$

$$\Rightarrow \sin^2\gamma = \cos^2\alpha + \cos^2\beta$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

(D) Let $\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, we get

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = 4\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$$

$$= 4\begin{vmatrix} x+1 & -3 \\ 2x-1 & 0 \end{vmatrix} = 24x - 12$$

$\therefore$ $A = 24$, $B = -12$

So, $A + 2B = 0$

56. (0) : We have, $\begin{vmatrix} 2 & 5 & 8 \\ a23 & b53 & c83 \\ a & b & c \end{vmatrix}$

$$= \begin{vmatrix} 2 & 5 & 8 \\ 100a+20+3 & 100b+50+3 & 100c+80+3 \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 & 8 \\ 100a & 100b & 100c \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2 & 5 & 8 \\ 20 & 50 & 80 \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2 & 5 & 8 \\ 3 & 3 & 3 \\ a & b & c \end{vmatrix}$$

$$= 100\begin{vmatrix} 2 & 5 & 8 \\ a & b & c \\ a & b & c \end{vmatrix} + 10\begin{vmatrix} 2 & 5 & 8 \\ 2 & 5 & 8 \\ a & b & c \end{vmatrix} + 3\begin{vmatrix} 2 & 5 & 8 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$= 0 + 0 + 3\begin{vmatrix} 2 & 3 & 6 \\ 1 & 0 & 0 \\ a & b-a & c-a \end{vmatrix}$$

(Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$)

$= -3[3(c - a) - 6(b - a)] = -9[c - a - 2b + 2a]$

$= -9(a - 2b + c) = 0$ [$\because$ a, b, c are in A.P., $\therefore$ $2b = a + c$]

57. (8) : adj $P = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 4 & 7 & 3 \end{bmatrix}$

$|\text{adj } P| = 4 \Rightarrow |P|^2 = 4 \Rightarrow |P| = \pm 2$

$\therefore$ Required sum $= (2)^2 + (-2)^2 = 8$

58. (1) : $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$$

Observing A, A^2, A^3 we can conclude that

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} \Rightarrow \det(A^n) = 1 - 4n^2 + 4n^2 = 1$$

$\therefore$ $\det(A^{2005}) = 1$

59. (1) : $\lambda = 1$, $f(-x) = -f(x) = g(x)$

$\therefore$ $f(x) \cdot g(x) = -(f(x))^2$

or $f(1)g(1) = -(f(1))^2 = -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix}^2 = -4 \Rightarrow \lambda = 1$

60. (7) : Here, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ -1 & 7-\lambda \end{vmatrix} = 0$$

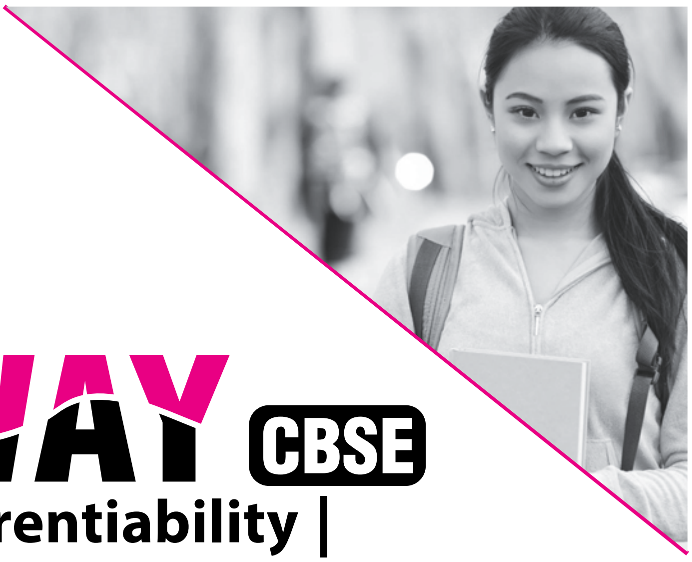
$\Rightarrow (1 - \lambda)(7 - \lambda) = 0 \Rightarrow \lambda^2 - 8\lambda + 7 = 0$

$\Rightarrow A^2 - 8A + 7I_2 = 0 \Rightarrow A^2 = 8A - 7I_2$

$\Rightarrow k = -7 \Rightarrow |k| = 7$

CLASS XII Series 4

ACE YOUR WAY CBSE



Continuity and Differentiability | Application of Derivatives

Important Formulae

CONTINUITY AND DIFFERENTIABILITY

- *A function $f(x)$ is said to be continuous at a point $x = a$ of its domain, if $\lim_{x \to a} f(x) = f(a)$.*
- *Let f and g be two real functions, continuous at $x = a$. Let α be a real number, then*
  - $f + g$ *is continuous at* $x = a$
  - $f - g$ *is continuous at* $x = a$
  - αf *is continuous at* $x = a$
  - fg *is continuous at* $x = a$
  - $\frac{1}{f}$ *is continuous at* $x = a$, $f(a) \neq 0$
  - $\frac{f}{g}$ *is continuous at* $x = a$, $g(a) \neq 0$
  - *A function f is said to be differentiable function if it is differentiable at every point of its domain.*
- *Following rules were established as part of algebra of derivatives :*
  - $(u \pm v)' = u' \pm v'$ *(Sum or difference)*
  - $(uv)' = u'v + uv'$ *(Product rule)*
  - $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$ *(Quotient rule)*
  - *If $y = f(t)$ and $t = g(x)$, then* $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ *(Composite function or chain rule)*
  - *If $x = f(t)$ and $y = g(t)$, then* $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
  - *If $y = f(x)$, then* $\frac{dy}{dx} = f'(x)$, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$ *(Second order derivative)*
- **Some General Derivatives**
  - $\frac{d}{dx}(x^n) = nx^{n-1}$
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
  - $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$
  - $\frac{d}{dx}(\sec x) = \sec x \tan x$
  - $\frac{d}{dx}(e^{ax}) = ae^{ax}$
  - $\frac{d}{dx}(e^x) = e^x$
  - $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$; $x \in (-1, 1)$
  - $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$; $x \in (-1, 1)$
  - $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$; $x \in R$
  - $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$; $x \in R$
  - $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$; $x \in R - [-1, 1]$
  - $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$; $x \in R - [-1, 1]$
  - $\frac{d}{dx}(\log_e x) = \frac{1}{x}$; $x > 0$
  - $\frac{d}{dx}(a^x) = a^x \log_e a$; $a > 0$
  - $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$; $x > 0$ and $a > 0$

- **Mean Value Theorems**
  - **Rolle's theorem :** *If $f(x)$: $[a,b] \to R$ is continuous on $[a, b]$, differentiable on (a,b), such that $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.*
  - **Geometrical meaning :** *The tangent at point c on the curve $y = f(x)$ is parallel to x-axis.*
  - **Lagrange's mean value theorem :** *If $f(x)$: $[a,b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b) then there exists at least one $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$*
  - **Geometrical meaning :** *The tangent at point c on the curve $y = f(x)$ is parallel to the chord joining end points.*

APPLICATION OF DERIVATIVES

- *The slope of the tangent to the curve $y = f(x)$ at (x_1, y_1) is $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$*
- *The slope of the normal to the curve $y = f(x)$ at (x_1, y_1) is $\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$.*
- *Slope of tangent, $m = \dfrac{dy}{dx} = \tan\theta$, θ is the angle made by tangent with +ve x-axis.*
- *If $\dfrac{dy}{dx} = 0$ at (x_1, y_1), then tangent is parallel to x-axis.*
- *If $\dfrac{dy}{dx} = \infty$ at (x_1, y_1), then tangent is perpendicular to x-axis.*
- *The equation of tangent at (x_1, y_1) is*
  $y - y_1 = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$ *and equation of normal at* (x_1, y_1) *is* $y - y_1 = \left(\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}\right)(x - x_1)$
- *If $y = f(x) \Rightarrow \delta y = f'(x)\,\delta x$, where $\delta y = f(x + \delta x) - f(x)$, then*
  - *δx is absolute error*
  - *$\dfrac{\delta x}{x}$ is relative error*
  - *$\left(\dfrac{\delta x}{x} \times 100\right)$ is percentage error*

INCREASING AND DECREASING FUNCTIONS (MONOTONICITY)

1. Without derivative test : *A function $f(x)$ in the open interval (a, b) is said to be*

- *Increasing function: If $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in (a, b)$*
- *Strictly increasing function: If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$*
- *Decreasing function: If $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in (a, b)$*
- *Strictly decreasing function: If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$*

2. First derivative test : *A function f is said to be continuous on $[a, b]$ and differentiable in (a, b), then f is*

- *Increasing function in $[a, b]$: If $f'(x) \ge 0$ for all $x \in (a, b)$*
- *Strictly increasing in (a, b): If $f'(x) > 0$ for all $x \in (a, b)$*
- *Decreasing function in $[a, b]$: If $f'(x) \le 0$ for all $x \in (a, b)$*
- *Strictly decreasing in (a, b): If $f'(x) < 0$ for all $x \in (a, b)$*

  Note: $f(x)$ is a constant function in $[a, b]$ if $f'(x) = 0$ for all $x \in (a, b)$

3. Critical or Stationary point: *The value of x for which $f'(x) = 0$.*

MAXIMA AND MINIMA

Maximum	*Let $f(x)$ be a function with domain $D \subset R$. $f(x)$ is said to attain the maximum value at a point $a \in D$, if $f(x) < f(a)$, for all $x \in D$. a is called the point of maxima and $f(a)$ is known as the maximum value or the greatest value of $f(x)$.*
Minimum	*Let $f(x)$ be a function with domain $D \subset R$. $f(x)$ is said to attain the minimum value at a point $a \in D$, if $f(x) > f(a)$, for all $x \in D$. a is called the point of minima and $f(a)$ is known as the minimum value or the least value of $f(x)$.*

Without derivative test

Local Maxima	*A function $f(x)$ is said to attain local maximum value at $x = a$, if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that $f(x) < f(a)$ for all $x \in (a - \delta, a + \delta)$, $x \ne a$. $f(a)$ is called the local maximum value of $f(x)$ at $x = a$.*
Local Minima	*A function $f(x)$ is said to attain local minimum value at $x = a$, if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that $f(x) > f(a)$ for all $x \in (a - \delta, a + \delta)$, $x \ne a$. $f(a)$ is called the local minimum value of $f(x)$ at $x = a$.*

First derivative test

- *Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I.*

Local maximum	*If $f'(x)$ changes sign from + ve to – ve then c is a point of local maxima.*
Local minimum	*If $f'(x)$ changes sign from – ve to + ve then c is a point of local minima.*
Point of inflexion	*If $f'(x)$ does not changes sign then c is neither point of local maxima nor a point of local minima. Such a point is called point of inflexion.*

Second derivative test

- *Let f be a function defined on an interval I. Let f be twice differentiable at c.*

Local maximum	*If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has local maxima at c and $f(c)$ is maximum value of $f(x)$.*
Local minimum	*If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has local minima at c and $f(c)$ is minimum value of $f(x)$.*
Test fails	*The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.*

Absolute maxima and Absolute minima

Absolute maxima and Absolute minima	*Let f be a differentiable function on a closed interval $[a, b]$, then it attains the absolute maximum (absolute minimum) at stationary points (points where $f'(x) = 0$) or at the end points of the inverval $[a, b]$.*

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Differentiate the following functions:
$$\left(\frac{2\tan x}{\tan x+\cos x}\right)^2$$

2. Find an angle θ, $0<\theta<\frac{\pi}{2}$, which increases twice as fast as $\sin\theta$.

3. Given $y = x^4 - 10$ and x changes from 2 to 1.99, what is the approximate change in y.

4. Determine whether the following function is strictly increasing or decreasing for the stated values of x:
$$f(x) = x+\frac{1}{x},\ x\ge 1$$

5. Find the maximum and the minimum values of the function : $f(x) = 2x^3 + 5$.

SHORT ANSWER TYPE

6. If $y = x^{e^{-x^2}}$, find $\frac{dy}{dx}$.

7. Discuss the continuity of the following function at $x=0$:
$$f(x)=\begin{cases}\dfrac{x^4+2x^3+x^2}{\tan^{-1}x}, & x\ne 0\\ 0\ , & x=0\end{cases}$$

8. Does the Lagrange's mean value theorem apply to $f(x) = x^{1/3}$, $-1 \le x \le 1$?

9. If $y=\sqrt{1+\sqrt{1+x^4}}$, prove that $y(y^2-1)\frac{dy}{dx}=x^3$.

10. The curve $3y^2 = 2ax^2 + 6b$ passes through the point $P(3, -1)$ and the gradient of the curve at P is -1. Find the values of a and b.

LONG ANSWER TYPE - I

11. Show that the function $f(x) = |x - 3|$, $x \in \boldsymbol{R}$ is continuous but not differentiable at $x = 3$.

12. Find the derivative of the following function:
$$\log\sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}$$

13. Find the local maxima and local minima of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$.

14. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that
$$(x^2+4)\left(\frac{dy}{dx}\right)^2=n^2(y^2+4)$$

15. Find the values of 'a' for which the function f defined as
$$f(x)=\begin{cases}a\sin\dfrac{\pi}{2}(x+1), & x\le 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}$$
is continuous at $x = 0$.

LONG ANSWER TYPE - II

16. If $(ax + b)\, e^{y/x} = x$, then show that
$$x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2.$$

17. The length x of a rectangle is decreasing at the rate of 3 cm/min and the width y is increasing at the rate of 2 cm/min. When $x = 10$ cm and $y = 6$ cm, find the rate of change of
(i) the perimeter
(ii) the area of the rectangle

18. Water is leaking from a conical funnel at the rate of 5 cm^3/sec. If the radius of the base of the funnel is 10 cm and its height is 20 cm, find the rate at which the water level is dropping when it is 5 cm from the top.

19. Find the values of x for which $f(x) = (x (x - 2))^2$ is an increasing function. Also, find the points on the curve, where the tangent is parallel to x-axis.

20. Verify Rolle's theorem for each of the following functions on the indicated intervals.
(i) $f(x) = x(x + 3)e^{-x/2}$ on $[-3, 0]$
(ii) $f(x) = e^x(\sin x - \cos x)$ on $[\pi/4, 5\pi/4]$.

SOLUTIONS

1. Let $y=\left(\frac{2\tan x}{\tan x+\cos x}\right)^2$;

Diff. w.r.t. x, we get
$$\frac{dy}{dx}=2\left(\frac{2\tan x}{\tan x+\cos x}\right)\cdot\frac{(\tan x+\cos x)\cdot 2\sec^2 x-2\tan x\cdot(\sec^2 x-\sin x)}{(\tan x+\cos x)^2}$$
$$=\frac{8\tan x\,(\cos x\sec^2 x+\tan x\sin x)}{(\tan x+\cos x)^3}$$
$$=\frac{8\tan x\,(\sec x+\tan x\sin x)}{(\tan x+\cos x)^3}$$

2. Let the radian measure of an angle at any time t be θ.
According to the question,

$$\frac{d}{dt}(\theta) = 2\frac{d}{dt}(\sin\theta)$$

$$\Rightarrow \frac{d\theta}{dt} = 2\cdot\cos\theta\frac{d\theta}{dt} \Rightarrow 1 = 2\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{2} \text{ but } 0 < \theta < \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{3}$$

3. Given $y = x^4 - 10$ ...(i)

Diff. (i), w.r.t. x, we get $\frac{dy}{dx} = 4x^3$.

As x changes from 2 to 1.99, $x = 2$ and $x + \delta x = 1.99$, then $\delta x = -0.01$,

When $x = 2$, $y = 2^4 - 10 = 16 - 10 = 6$

For $x = 2$ and $\delta x = -0.01$

So, $\delta y = \frac{dy}{dx}\delta x = 4x^3\,\delta x = 4 \times 2^3 \times (-0.01)$

$= -32 \times 0.01 = -0.32$

The approximate change in $y = -0.32$ and $y + \delta y = 6 - 0.32 = 5.68$.

Hence, y changes from 6 to 5.68.

4. Given $f(x) = x + \frac{1}{x}$, $x \ge 1 \Rightarrow f'(x) = 1 - \frac{1}{x^2}$

Now $x \ge 1 \Rightarrow x^2 \ge 1 \Rightarrow \frac{1}{x^2} \le 1 \Rightarrow 1 - \frac{1}{x^2} \ge 0$

$\therefore f'(x) \ge 0$ for $x \ge 1$

Therefore, $f(x)$ is strictly increasing for $x \ge 1$.

5. Given $f(x) = 2x^3 + 5$, $D_f = R$.

Now when $x \to \infty$, $f(x) \to \infty$ and when $x \to -\infty$, $f(x) \to -\infty$

Therefore, f has neither maxima nor minima.

In fact, f is a strictly increasing function for all $x \in R$.

6. Given $y = x^{e^{-x^2}}$

Taking logarithm on both sides, we get

$\log y = e^{-x^2}\log x$,

Differentiate w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = e^{-x^2}\cdot\frac{1}{x} + \log x\cdot e^{-x^2}\cdot(-2x)$$

$$\Rightarrow \frac{dy}{dx} = y\cdot e^{-x^2}\left(\frac{1}{x} - 2x\log x\right)$$

$$\Rightarrow \frac{dy}{dx} = x^{e^{-x^2}}\cdot e^{-x^2}\left(\frac{1}{x} - 2x\log x\right)$$

7. Here $f(0) = 0$

$$\underset{x\to 0}{\text{Lt}} f(x) = \underset{x\to 0}{\text{Lt}}\frac{x^4 + 2x^3 + x^2}{\tan^{-1}x} = \underset{x\to 0}{\text{Lt}}\frac{x^2(x^2 + 2x + 1)}{\tan^{-1}x}$$

$$= \underset{x\to 0}{\text{Lt}}\, x(x^2 + 2x + 1)\cdot\underset{x\to 0}{\text{Lt}}\frac{x}{\tan^{-1}x}$$

$$\left[\because \underset{t\to 0}{\text{Lt}}\frac{\tan t}{t} = 1\right]$$

$= 0\cdot(0 + 0 + 1)\cdot 1 = 0$

[Put $\tan^{-1}x = t \Rightarrow x = \tan t$, when $x \to 0$, $t \to 0$]

$\Rightarrow \underset{x\to 0}{\text{Lt}} f(x) = 0 = f(0) \Rightarrow f$ is continuous at $x = 0$.

8. Given, $f(x) = x^{1/3}$, $x \in [-1, 1]$ ...(i)

(1) $f(x)$ is continuous in $[-1, 1]$.

(2) Differentiating (i) w.r.t. x, we get

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}},\ x \ne 0 \qquad \text{...(ii)}$$

$\Rightarrow$ The derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow f(x)$ is not derivable in $(-1, 1)$.

Thus, the condition of Lagrange's mean value theorem is not satisfied by the function $f(x) = x^{1/3}$ in $[-1, 1]$ and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in $[-1, 1]$.

9. Given $y = \sqrt{1 + \sqrt{1 + x^4}}$

$$\Rightarrow y^2 = 1 + \sqrt{1 + x^4} \Rightarrow y^2 - 1 = \sqrt{1 + x^4}$$

$\Rightarrow (y^2 - 1)^2 = 1 + x^4$

Differentiating w.r.t. x, we get

$$2(y^2 - 1)\left(2y\frac{dy}{dx} - 0\right) = 0 + 4x^3$$

$$\Rightarrow 4y(y^2 - 1)\frac{dy}{dx} = 4x^3 \Rightarrow y(y^2 - 1)\frac{dy}{dx} = x^3$$

10. The given curve is $3y^2 = 2ax^2 + 6b$ ...(i)

Differentiating (i) w.r.t. x, we get

$$3\cdot 2y\frac{dy}{dx} = 2a\cdot 2x + 0 \Rightarrow \frac{dy}{dx} = \frac{2ax}{3y}$$

$\therefore$ The gradient of the curve at $P(3, -1)$ = Slope of tangent at $P = \frac{2a\cdot 3}{3\cdot(-1)} = -2a$.

But gradient of curve at $P = -1$ (given)

$$\Rightarrow -2a = -1 \Rightarrow a = \frac{1}{2}$$

Also the curve (i) passes through the point $P(3, -1)$,

$\therefore 3\cdot(-1)^2 = 2a\cdot 3^2 + 6b \Rightarrow 3 = 18a + 6b$

$$\Rightarrow 2b = 1 - 6a = 1 - 6\cdot\frac{1}{2} = -2 \Rightarrow b = -1.$$

Hence, $a = \frac{1}{2}$ and $b = -1$.

11. Given, $f(x) = |x - 3|$, $x \in R$.

$\therefore f(3) = |3 - 3| = |0| = 0$

Now, $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} |x-3|$

$$= \lim\limits_{x \to 3^-} (-(x-3)) = -(3-3) = 0$$

and $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} |x-3|$

$$= \lim\limits_{x \to 3^+} (x-3) = 3-3 = 0$$

Thus $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = f(3)$

$\Rightarrow$ f is continuous at $x = 3$

L.H.D. $= f'(3) = \lim\limits_{h \to 0^-} \dfrac{f(3+h)-f(3)}{h}$

$$= \lim\limits_{h \to 0^-} \frac{|3+h-3|-0}{h} = \lim\limits_{h \to 0^-} \frac{|h|}{h} = \lim\limits_{h \to 0^-} \frac{-h}{h} = -1$$

R.H.D. $= f'(3) = \lim\limits_{h \to 0^+} \dfrac{f(3+h)-f(3)}{h}$

$$= \lim\limits_{h \to 0^+} \frac{|3+h-3|-0}{h} = \lim\limits_{h \to 0^+} \frac{|h|}{h} = \lim\limits_{h \to 0^+} \frac{h}{h} = 1$$

$\Rightarrow$ L.H.D. $\neq$ R.H.D.

$\Rightarrow$ f is not differentiable at $x = 3$.

12. Let $y = \log \sqrt{\dfrac{1+\cos^2 x}{1-e^{2x}}}$

$$= \frac{1}{2}[\log(1+\cos^2 x) - \log(1-e^{2x})]$$

Diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+\cos^2 x}.\frac{d}{dx}(1+\cos^2 x) - \frac{1}{1-e^{2x}}.\frac{d}{dx}(1-e^{2x})\right]$$

$$= \frac{1}{2}\left[\frac{2\cos x}{1+\cos^2 x}\cdot(-\sin x) + \frac{e^{2x}}{1-e^{2x}}\cdot 2\right]$$

$$= -\frac{\sin x \cos x}{1+\cos^2 x} + \frac{e^{2x}}{1-e^{2x}}.$$

13. Given, $f(x) = \sin x - \cos x$, $0 < x < 2\pi$.

The function f is differentiable for all $x \in (0, 2\pi)$

$f'(x) = \cos x + \sin x$ and $f''(x) = -\sin x + \cos x$.

Now, $f'(x) = 0 \Rightarrow \cos x + \sin x = 0 \Rightarrow \tan x = -1$

but $x \in (0, 2\pi) \Rightarrow x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

Therefore, the points where extremum may occur are $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$$

$\Rightarrow$ f has local maxima at $x = \dfrac{3\pi}{4}$.

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= -\left(-\sin\frac{\pi}{4}\right) + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0$$

$\Rightarrow$ f has local minima at $x = \dfrac{7\pi}{4}$.

14. Given, $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$

$$\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \sec\theta\tan\theta + \frac{\sin\theta}{\cos\theta}.\cos\theta$$

$$= \tan\theta(\sec\theta + \cos\theta)$$

and $\dfrac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta - n\cos^{n-1}\theta(-\sin\theta)$

$$= n[\sec^n\theta\tan\theta + \cos^{n-1}\theta\cdot\frac{\sin\theta}{\cos\theta}\cdot\cos\theta]$$

$$= n\tan\theta(\sec^n\theta + \cos^n\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$= \frac{n(\sec^n\theta + \cos^n\theta)}{\sec\theta + \cos\theta}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = n^2\frac{(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$= n^2\frac{(\sec^n\theta - \cos^n\theta)^2 + 4\sec^n\theta\cos^n\theta}{(\sec\theta - \cos\theta)^2 + 4\sec\theta\cos\theta}$$

$$= n^2\cdot\frac{y^2+4}{x^2+4} \Rightarrow (x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

15. Here, $f(0) = a\sin\dfrac{\pi}{2}(0+1) = a\sin\dfrac{\pi}{2} = a \times 1 = a$

$$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} a\sin\frac{\pi}{2}(x+1)$$

$$= a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a \times 1 = a$$

and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{\tan x - \sin x}{x^3}$

$$= \lim\limits_{x \to 0^+} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim\limits_{x \to 0^+} \frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$= \lim\limits_{x \to 0^+} \frac{\sin x(1-\cos x)}{x^3\cos x} \times \frac{1+\cos x}{1+\cos x}$$

$$= \underset{x \to 0^+}{\text{Lt}} \frac{\sin x \cdot \sin^2 x}{x^3 \cos x(1 + \cos x)}$$

$$= \left(\underset{x \to 0^+}{\text{Lt}} \frac{\sin x}{x} \right)^3 \underset{x \to 0^+}{\text{Lt}} \frac{1}{\cos x\,(1 + \cos x)}$$

$$= 1^3 \cdot \frac{1}{1(1+1)} = \frac{1}{2}$$

For the function f to be continuous at $x = 0$, we must have $\underset{x \to 0^-}{\text{Lt}} f(x) = \underset{x \to 0^+}{\text{Lt}} f(x) = f(0)$

$$\Rightarrow a = \frac{1}{2}$$

16. Given $(ax + b)e^{y/x} = x$

$$\Rightarrow e^{y/x} = \frac{x}{ax + b}$$

Taking log on both sides, we get

$$\frac{y}{x} \cdot \log e = \log \frac{x}{ax + b}$$

$$\Rightarrow \frac{y}{x} = \log x - \log(ax + b) \qquad (\because \log e = 1)$$

Differentiating w.r.t. x, we get

$$\frac{x \cdot \frac{dy}{dx} - y \cdot 1}{x^2} = \frac{1}{x} - \frac{1}{ax + b} \cdot a$$

$$\Rightarrow x\frac{dy}{dx} - y = x^2 \cdot \frac{ax + b - ax}{x(ax + b)}$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{bx}{ax + b} \qquad \text{...(i)}$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 - \frac{dy}{dx} = \frac{(ax + b) \cdot b - bx \cdot a}{(ax + b)^2}$$

$$\Rightarrow x\frac{d^2y}{dx^2} = \frac{b^2}{(ax + b)^2} \quad \Rightarrow x^3\frac{d^2y}{dx^2} = \left(\frac{bx}{ax + b}\right)^2$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2 \qquad \text{(using (i))}$$

17. Since, the length x of a rectangle is decreasing and the width y is increasing, we have

$$\frac{dx}{dt} = -3 \text{ cm/min and } \frac{dy}{dt} = 2 \text{ cm/min}$$

(i) Perimeter $(P) = 2(x + y)$,
Diff. w.r.t. t, we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3 + 2) \text{ cm/min} = -2 \text{ cm/min}$$

When $x = 10$ cm, $y = 6$ cm, $\frac{dP}{dt} = -2$ cm/min

Hence, the perimeter is decreasing at the rate of 2 cm/min.

(ii) Area $(A) = xy$, Diff. w.r.t. t, we get

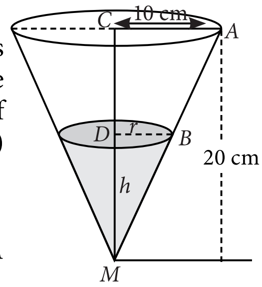
$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = (x \cdot (2) + y(-3)) \text{ cm}^2\text{/min}$$

When $x = 10$ cm, $y = 6$ cm,

$$\frac{dA}{dt} = (10 \cdot (2) + 6 \cdot (-3)) \text{ cm}^2\text{/min} = 2 \text{ cm}^2\text{/min}$$

Hence, the area is increasing at the rate of 2 cm^2/min.

18. At any time t, water forms a cone. Let r cm be the radius of this cone and h cm be its height, then volume of water in the funnel (V) $= \frac{1}{3}\pi r^2 h$



Since Δs DMB and CMA are similar,

$$\therefore \frac{DM}{CM} = \frac{DB}{CA} \Rightarrow \frac{h}{20} = \frac{r}{10} \Rightarrow r = \frac{h}{2}$$

$\therefore$ Volume of water at any time (V)

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Diff. w.r.t. t, we get $\frac{dV}{dt} = \frac{\pi}{12}\left(3h^2\right)\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$

Since water is leaking from the conical funnel at the rate of 5 cm^3/sec.

$$\therefore \frac{dV}{dt} = -5$$

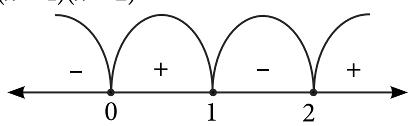
$$\Rightarrow -5 = \frac{\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{20}{\pi h^2}$$

When the water level is 5 cm from top,
$h = 20 - 5 = 15$ cm, then

$$\frac{dh}{dt} = -\frac{20}{\pi \times (15)^2} = -\frac{4}{45\pi} \text{ cm/sec}$$

Hence, the rate at which the water level is dropping $= \frac{4}{45\pi}$ cm/sec.

19. Given $f(x) = (x(x - 2))^2 = x^2(x - 2)^2$, $D_f = R$.
Differentiating w.r.t., x, we get
$f'(x) = x^2 \cdot 2(x - 2) + (x - 2)^2 \cdot 2x$
$= 2x(x - 2)(x + x - 2) = 2x(x - 2)(2x - 2)$
$= 4x(x - 1)(x - 2)$



Now, the given function f is (strictly) increasing iff $f'(x) > 0$

$\Rightarrow$ $x \in (0, 1) \cup (2, \infty)$

Further, the tangents will be parallel to x-axis iff $f'(x) = 0$

$\Rightarrow$ $x = 0, 1, 2$

The given curve is $y = x^2(x - 2)^2$

When $x = 0$, $y = 0$;

When $x = 1$, $y = 1^2(1 - 2)^2 = 1 \times (-1)^2 = 1 \times 1 = 1$;

When $x = 2$, $y = 2^2 (2 - 2)^2 = 4 \times 0 = 0$.

$\therefore$ The points on the given curve where the tangents are parallel to x-axis are (0, 0), (1, 1) and (2, 0).

20. (i) Since a polynomial function and an exponential function are everywhere continuous and differentiable. Therefore, $f(x)$, being product of these two, is continuous on $[-3, 0]$ and differentiable on $(-3, 0)$.

Also, $f(-3) = -3\ (-3 + 3)\ e^{3/2} = 0$ and $f(0) = 0$

$\therefore$ $f(-3) = f(0)$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[-3, 0]$.

Consequently, there exists $c \in (-3, 0)$ such that $f'(c) = 0$.

Now, $f(x) = x(x + 3)e^{-x/2}$

$\Rightarrow$ $f'(x) = (2x + 3)\ e^{-x/2} + (x^2 + 3x)\ (-1/2)e^{-x/2}$

$$= e^{-x/2}\left\{\frac{-x^2+x+6}{2}\right\}$$

$\therefore$ $f'(x) = 0$

$$\Rightarrow e^{-x/2}\left\{\frac{-x^2+x+6}{2}\right\}=0 \Rightarrow -x^2 + x + 6 = 0$$

$\Rightarrow$ $x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2, 3$

Thus, $c = -2 \in (-3, 0)$ such that $f'(c) = 0$.

(ii) Since exponential function and sine and cosine functions are everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on $[\pi/4, 5\pi/4]$ and differentiable on $(\pi/4, 5\pi/4)$.

Also,

$$f\left(\frac{\pi}{4}\right)=e^{\pi/4}\left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)=e^{\pi/4}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)=0$$

$$f\left(\frac{5\pi}{4}\right)=e^{5\pi/4}\left(\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}\right)$$

$$=e^{5\pi/4}\left(-\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{5\pi}{4}\right)=e^{5\pi/4}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=0$$

$$\therefore f\left(\frac{\pi}{4}\right)=f\left(\frac{5\pi}{4}\right)$$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[\pi/4, 5\pi/4]$.

Consequently, there exists $c \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$

Now, $f(x) = e^x (\sin x - \cos x)$

$\Rightarrow$ $f'(x) = e^x (\sin x - \cos x) + e^x (\cos x + \sin x)$

$= 2e^x \sin x$

$\therefore$ $f'(x) = 0 \Rightarrow 2e^x \sin x = 0$

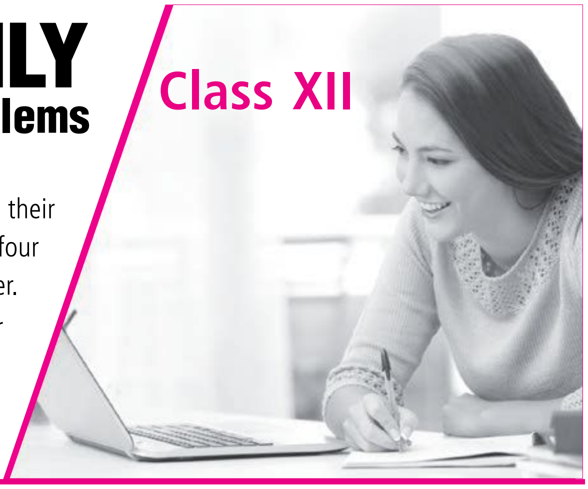
$\Rightarrow$ $\sin x = 0$

$\Rightarrow$ $x = \pi$

Thus, $c = \pi \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$.

MPP-4 MONTHLY Practice Problems

Class XII



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Continuity and Differentiability

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. For a real number y, let $[y]$ denotes the greatest integer less than or equal to y. Let $f(x)=\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}$, then

(a) $f(x)$ is discontinuous at some x

(b) $f(x)$ is continuous at all x, but the derivative $f'(x)$ does not exist for some x

(c) $f'(x)$ exists for all x, but the derivative $f'(x_0)$ does not exist for some x

(d) $f(x)$ exists for all x

2. Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$

(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

3. If $f(x)$ is a continuous and differentiable function and $f(1/n) = 0 \ \forall \ n \geq 1$ and $n \in I$, then

(a) $f(x) = 0, x \in (0, 1]$

(b) $f(0) = 0, f'(0) = 0$

(c) $f(0) = 0 = f'(0), x \in (0, 1]$

(d) $f(0) = 0$ and $f'(0)$ need not be zero

4. The value of p and q for which the function

$$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x} & , x<0\\ q & , x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}} & , x>0\end{cases}$$

is continuous for all x in R, are

(a) $p=\frac{1}{2}, q=\frac{3}{2}$ (b) $p=\frac{1}{2}, q=-\frac{3}{2}$

(c) $p=\frac{5}{2}, q=\frac{1}{2}$ (d) $p=-\frac{3}{2}, q=\frac{1}{2}$

5. If number of points of discontinuity of the function $f(x) = [2 + 10 \sin x]$, in $x \in \left[0, \frac{\pi}{2}\right]$ is same as number of points of non-differentiability of the function $g(x) = (x - 1)(x - 2)/(x - 1)(x - 2) ..... (x - 2m)$, $(m \in N)$ in $x \in (-\infty, \infty)$ then the value of m is

(a) 5 (b) 6 (c) 7 (d) 8

6. If $f : (0, 2) \to R$ is defined as

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 2x-1, & \text{if } x \text{ is irrational}\end{cases} \text{ then}$$

(a) f is differentiable exactly at one point.

(b) f is differentiable exactly at two points.

(c) f is not differentiable at any point in (0,2).

(d) f is differentiable at every point in (0, 2).

One or More Than One Option(s) Correct Type

7. If $x + |y| = 2y$, then y as function of x is

(a) defined for all real x

(b) continuous at $x = 0$

(c) differentiable for all x

(d) such that $\dfrac{dy}{dx}=\dfrac{1}{3}$ for $x < 0$

8. Let $g(x) = xf(x)$, where $f(x)=\begin{cases}x\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$. At $x = 0$, then

(a) g is differentiable but g' is not continuous
(b) g is differentiable while f is not
(c) both f and g are differentiable
(d) g is differentiable and g' is continuous

9. For every integer n, let a_n and b_n be real numbers. Let function $f: R \to R$ be given by

$$f(x)=\begin{cases} a_n+\sin \pi x, & \text{for } x\in[2n,2n+1] \\ b_n+\cos \pi x, & \text{for } x\in(2n-1,2n) \end{cases},$$

for all integers n. If f is continuous, then which of the following hold(s) for all n?
(a) $a_{n-1}-b_{n-1}=0$ (b) $a_n-b_n=1$
(c) $a_n-b_{n+1}=1$ (d) $a_{n-1}-b_n=-1$

10. The function

$$g(x)=\begin{cases} \dfrac{1-a^x+xa^x\log_e a}{a^x x^2}; & x<0 \\ \dfrac{2^x a^x-x\log_e 2-x\log_e a-1}{x^2}; & x>0 \end{cases}, a>0$$

is continuous at $x = 0$, then
(a) the value of a is $\dfrac{1}{\sqrt{2}}$
(b) the value of a is $\dfrac{1}{2}$
(c) the value of $g(0)$ is $\dfrac{(\log_e 2)^2}{8}$
(d) the value of $g(0)$ is $\dfrac{(\log_e 2)^2}{4}$

11. Which of the following functions is/are thrice differentiable at $x = 0$?
(a) $f(x) = |x^3|$ (b) $f(x) = x^3\,|x|$
(c) $f(x) = |x|\sin^3 x$ (d) $f(x) = x\,|\tan^3 x|$

12. f is a continuous function in $[a, b]$ and g is a continuous function in $[b, c]$. A function $h(x)$ is defined as

$$h(x)=\begin{cases} f(x), & x\in[a,b) \\ g(x), & x\in(b,c] \end{cases}$$

Now, if $f(b) = g(b)$, then
(a) $h(x)$ may or may not be continuous in $[a, c]$
(b) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$
(c) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$
(d) $h(x)$ has a removable discontinuity at $x = b$

13. Let $f : R \to R$ be a function such that $f(x+y)=f(x)+f(y)$, $\forall\, x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then
(a) $f(x)$ is differentiable only in a finite interval containing zero
(b) $f(x)$ is continuous $\forall\, x \in R$
(c) $f'(x)$ is constant $\forall\, x \in R$
(d) $f(x)$ is differentiable except at finitely many points

Comprehension Type

Consider two functions $y = f(x)$ and $y = g(x)$ defined as

$$f(x)=\begin{cases} ax^2+b; & 0\le x\le 1 \\ bx+2b; & 1<x\le 3 \\ (a-1)x+2c-3; & 3<x\le 4 \end{cases}$$

and
$$g(x)=\begin{cases} cx+d; & 0\le x\le 2 \\ ax+3-c; & 2<x<3 \\ x^2+b+1; & 3\le x\le 4 \end{cases}$$

14. Let f be differentiable at $x = 1$ and $g(x)$ be continuous at $x = 3$. If the roots of the quadratic equation $x^2 + (a + b + c)\,\alpha x + 49(k + k\alpha) = 0$ are real and distinct for all values of α then possible values of k will be
(a) $k \in (-1, 0)$ (b) $k \in (\infty, 0)$
(c) $k \in (1, 5)$ (d) $k \in (-1, 1)$

15. $\lim\limits_{x\to 2}\dfrac{f(x)}{|g(x)|+1}$ exists and f is differentiable at $x = 1$. The value of limit will be
(a) -2 (b) -1 (c) 0 (d) 2

Matrix Match Type

16. Match the following :

	Column-I		Column-II
P.	$f(x)=\begin{cases} \log_e(1+x^3)\cdot\sin\dfrac{1}{x}, & x>0 \\ 0, & x\le 0 \end{cases}$	1.	discontinuous at $x = 0$
Q.	$g(x)=\begin{cases} (\log_e(1+x))^2\cdot\sin\dfrac{1}{x}, & x>0 \\ 0, & x\le 0 \end{cases}$	2.	continuous but non-differentiable at $x = 0$

R.	$h(x)=\begin{cases}\log_e\left(1+\dfrac{\sin x}{2}\right), & x>0\\ 1, & x\le 0\end{cases}$	3.	continuous and differentiable at $x = 0$
S.	$j(x)=\begin{cases}e^{-1/x^2}, & x>0\\ 0, & x\le 0\end{cases}$		

	P	Q	R	S
(a)	3	2	1	3
(b)	2	2	3	1
(c)	3	1	3	2
(d)	1	2	3	1

Integer Answer Type

17. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim\limits_{h\to 0}\dfrac{1}{h}f(1+h) = 5$, then $f'(1)$ equals

18. Number of points where $f(x) = |x\ \text{sgn}\ (1 - x^2)|$ is non-differentiable is

19. Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(2) = 9$, then $f(1.5) =$

20. Number of points where $f(x) = |\cos|x|| + \cos^{-1}(\text{sgn}\ x) + |\log_e x|$ is not differentiable in $(0, 2\pi)$ is

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SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 176

JEE MAIN

1. Let $0 < a < b < 125$. The number of pairs of integers (a, b), such that the A.M. of a and b exceeds their G.M. by 2, is

(a) 8 (b) 9 (c) 10 (d) 7

2. The sum of first 9 terms of the series

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots \text{ is}$$

(a) 142 (b) 192 (c) 71 (d) 96

3. Let a and b be integers, $0 < b < a$. The image of the point $A(a, b)$ in the line $y = x$ is B, the image of B in the y-axis is C, the image of C in the x-axis in D and the image of D in the y-axis is E. If the area of the pentagon is 451, then $a - b =$

(a) 1 (b) 2 (c) 3 (d) 5

4. The orthogonal trajectories of the family of parabolas $y^2 = 4ax$, $a \in R$, are conics of eccentricity

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{2}$

5. Two sides of a triangle are $\sqrt{6}$ and 4. The angle opposite to the smaller side is 30°. The area of the triangle is

(a) $2\sqrt{3}$ (b) $3\sqrt{2}$

(c) $2\sqrt{3}+\sqrt{2}$ (d) None of these.

JEE ADVANCED

6. $$\left(\frac{1}{4}\right)^{\frac{x-1}{2x^2+x-1}} < \left(\frac{1}{8}\right)^{\frac{1}{3x}} \text{ if } x =$$

(a) $\frac{5}{6}$ (b) $-\frac{5}{6}$ (c) $\frac{5}{12}$ (d) $-\frac{1}{6}$

COMPREHENSION

Let z_1 be a complex number of magnitude unity and z_2 be a complex number given by $z_2 = z_1^2 - z_1$.

7. If $\arg z_1 = \theta$, then $|z_2|$ is equal to

(a) $2\left|\sin\frac{\theta}{2}\right|$ (b) $2\left|\cos\frac{\theta}{2}\right|$

(c) $\sqrt{2}\left|\sin\frac{\theta}{2}\right|$ (d) $\sqrt{2}\left|\cos\frac{\theta}{2}\right|$

8. If $\arg z_1 = \theta$ and $4n\pi < \theta < (4n + 2)\pi$ (n is an integer), then $\arg z_2$ is equal to

(a) $\frac{3\theta}{2}$ (b) $\frac{\pi-3\theta}{2}$ (c) $\frac{\pi+3\theta}{2}$ (d) $\frac{\pi+\theta}{2}$

INTEGER TYPE

9. If $m = 2^{2013}$, then the last digit of the natural number $(2^m - 1)$ is

MATRIX MATCH

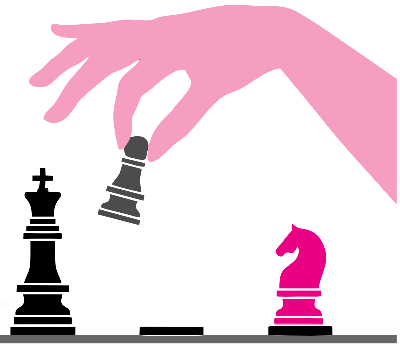
10. Match the following columns.

	Column I	Column II	
(P)	If $f(x)=x^{\alpha}\sin\left(\frac{1}{x}\right), x \neq 0$, $f(0) = 0$, then $f'(x)$ is continuous for $\alpha =$	1.	0
(Q)	If the curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $x = y$ at the origin, then $y(-1) =$	2.	4
(R)	If $f(x)=\sin x+\int_0^{\pi/2}\sin x\cos t\,dt$, then $2\int_0^{\pi/2} f(x)dx =$	3.	2
(S)	The area bounded by the common tangents to the curves $x^2 = 2(y^2 + 1)$ and $y^2 = 2(x^2 + 1)$ is	4.	3

	P	Q	R	S
(a)	3	2	1	4
(b)	4	1	2	3
(c)	2	3	1	4
(d)	4	1	3	2

See Solution Set of Maths Musing 175 on page no. 85

Challenging PROBLEMS ON CALCULUS



1. For $0 < a < b$, $\displaystyle\int_0^1 \frac{x^b - x^a}{\log x}dx =$

(a) $\log\left(\frac{1+a}{1+b}\right)$ (b) $\log\left(\frac{1+b}{1+a}\right)$

(c) $\log\left(\frac{1-a}{1-b}\right)$ (d) $\log\left(\frac{1-b}{1-a}\right)$

2. If P and Q are polynomials positive on R^+ and $k > 1$ then $\displaystyle\lim_{x\to\infty} \frac{1}{x^k}\left(\int_0^x \log\frac{P(t)}{Q(t)}dt\right) =$

(a) 0 (b) 1

(c) –1 (d) none of these

3. For $n \in N$, $\displaystyle\int_0^1 (-\log x)^n dx =$

(a) zero (b) 1

(c) $(n - 1)!$ (d) $n!$

4. Suppose f is continuous on $[a, b]$ and $f(a) = f(b) = 0$ and $\displaystyle\int_a^b f^2(x)dx = 1$ then the minimum value of $\displaystyle\int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x)dx$ is

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

5. Given $A = \left\{f : C[0,1] : \displaystyle\int_0^1 f(x)dx = 1\right\}$ then minimum value of $\displaystyle\int_0^1 (1+x^2)f^2(x)dx, f \in A$ is

(a) $\frac{2}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

6. f is continuous and strictly increasing on $[0, c]$ and $f(0) = 0$ then for $x \in [0, c]$, $\displaystyle\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt =$

(a) x (b) $f(x)$ (c) $xf(x)$ (d) $x - f(x)$

7. Given $\displaystyle\int_0^1 f(x)dx = 3, \int_0^1 xf(x)dx = 2$ then minimum value of $\displaystyle\int_0^1 f^2(x)dx$ is

(a) 4 (b) 8 (c) 12 (d) 24

8. Let f be continuous on $[0, 1]$ and $f(0) = 0 = f(1)$, $f'(0) = a$ then minimum value of $\displaystyle\int_0^1 (f''(x))^2 dx$ is

(a) a^2 (b) $2a^2$ (c) $3a^2$ (d) $4a^2$

9. Given that $\displaystyle\int \frac{a\cos x + b\sin x + c}{(1 - e\cos x)^2}dx$ is a rational function of $\sin x$ and $\cos x$ then

(a) $ae + c = 0$ (b) $a + ce = 0$

(c) $ab = ce$ (d) $ac = be$

10. Consider the function $f(x) = \begin{cases} (3x^2 - 1)\cos^{-1} x, x > 0 \\ \dfrac{x-k}{x^2-1}, x \le 0, k \in R \end{cases}$

If f has primitive functions in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ then $k =$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{-\pi}{2}$ (d) π

11. $\displaystyle\int \frac{2\sin x + 1}{(\sin x + 2)^2}dx =$

(a) $\dfrac{\cos x}{\sin x + 2} + c$ (b) $\dfrac{-\cos x}{\sin x + 2} + c$

(c) $\dfrac{\sin x}{\sin x + 2} + c$ (d) $\dfrac{-\sin x}{\sin x + 2} + c$

By : Tapas Kr. Yogi, Visakhapatnam Mob : 09533632105

12. If $\int \frac{(2x+3)dx}{x(x+1)(x+2)(x+3)+1} = \frac{-1}{g(x)} + c$ then the sum of the roots of $g(x) = 0$ is

(a) -1 (b) 1 (c) 3 (d) -3

13. If $\int \frac{1-(\cot x)^{19}}{\tan x + (\cot x)^{20}} dx = \frac{1}{k}\log|\sin^k x + \cos^k x| + c$

then $k =$

(a) 19 (b) 20 (c) 21 (d) 22

14. If $f(x)$ is monotonic and differentiable $a, b \in R$ and

$$I_1 = \int_a^b [f^2(x) - f^2(a)]dx, I_2 = \int_{f(a)}^{f(b)} x[b - f^{-1}(x)]dx,$$

then

(a) $I_1 = 2I_2$ (b) $2I_1 = I_2$
(c) $I_1 = I_2$ (d) $I_1 + I_2 = 0$

15. If $\int (1+2x^2)e^{x^2} dx = g(x) + c$ then $g(0) =$

(a) 0 (b) 1 (c) e^2 (d) e

16. $f: [0, 2] \to R$,

$$f(x) = \sqrt{x^3 + 2 - 2\sqrt{x^3+1}} + \sqrt{x^3 + 10 - 6\sqrt{x^3+1}},$$

$\int f(x)dx =$

(a) $2x + c$ (b) $x^2 + c$
(c) $x^3 + c$ (d) $x^3 - x^2 + c$

17. $I = \int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1}\, dx =$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) -1 (d) 0

18. $f: [0, 1] \to R$ is a continuous function such that

$\int_0^1 f(x)dx = \frac{1}{3} + \int_0^1 f^2(x^2)dx$ then $f\left(\frac{1}{2}\right) =$

(a) 1 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

SOLUTIONS

1. (b) : Since, $\lim_{x\to 0^+} \frac{x^b - x^a}{\log x} = 0$ and

$\lim_{x\to 1^-} \frac{x^b - x^a}{\log x} = b - a$, the given integral exists

Now, given integral $= \int_0^1 \left(\int_a^b x^t dt\right) dx = \int_a^b \left(\int_0^1 x^t dx\right) dt$

$$= \int_a^b \frac{x^{t+1}}{t+1}\Bigg|_0^1 dt = \int_a^b \left(\frac{1}{t+1}\right) dt = \log\left(\frac{1+b}{1+a}\right)$$

2. (a) : $\lim_{x\to\infty} \left(\frac{1}{x^k}\int_0^x \log\frac{P(t)}{Q(t)} dt\right)$

$$= \lim_{x\to\infty} \frac{\log P(x) - \log Q(x)}{kx^{k-1}} = 0$$

[By using L' Hospital's Rule]

3. (d) : $I_n = \int_0^1 (-\log x)^n dx = n \cdot \int_0^1 (-\log x)^{n-1} dx$

[Integration by parts]

$\Rightarrow$ $I_n = nI_{n-1}$

and since, $I_1 = 1$ so, $I_n = n!$

4. (c) : Schwartz inequality :

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$$

From the given integral values, we have

$$\int_a^b xf(x)f'(x)\,dx = \frac{1}{2}\int_a^b x(f^2(x))'\,dx$$

$$= \frac{1}{2}\left(xf^2(x)\Big|_a^b - \int_a^b f^2(x)dx\right) = \frac{-1}{2}$$

Hence, by Schwartz inequality

$$\frac{1}{4} \le \int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x)dx$$

5. (b) : By Schwartz inequality,

$$1 = \left|\int_0^1 f(x)dx\right| = \left|\int_0^1 \frac{f(x)\cdot\sqrt{1+x^2}}{\sqrt{1+x^2}}dx\right|$$

$$\Rightarrow\ 1 \le \sqrt{\int_0^1 (1+x^2)f^2(x)dx} \cdot \sqrt{\int_0^1 \frac{1}{1+x^2}dx}$$

$$i.e.,\ 1 \le \sqrt{\int_0^1 (1+x^2)f^2(x)dx} \cdot \frac{\sqrt{\pi}}{2}$$

Hence, required minimum is $\left(\frac{2}{\sqrt{\pi}}\right)^2 = \frac{4}{\pi}$ which is attained for $f(x) = \frac{4}{\pi(1+x^2)}$

6. (c) : Putting, $u = f^{-1}(t)$

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = \int_0^x f(t)dt + \int_0^x uf'(u)du$$

$= xf(x)$ [Integration by parts]

7. (c) : From Schwartz inequality :

$$\left(\int_0^1 f(x)\cdot(x+\lambda)dx\right)^2 \le \int_0^1 f^2(x)dx\cdot\int_0^1 (x+\lambda)^2 dx$$

$$\Rightarrow \quad (2+3\lambda)^2 \le \int_0^1 f^2(x)dx\cdot\frac{1}{3}(3\lambda^2+3\lambda+1)$$

$$i.e., \int_0^1 f^2(x)dx \ge \max\left[\frac{3(2+3\lambda)^2}{3\lambda^2+3\lambda+1}\right]=12$$

which is attained actually at $f(x) = 6x$.

8. (c): By Schwartz inequality

$$\left(\int_0^1 (1-x)f''(x)dx\right)^2 \le \int_0^1 (1-x)^2dx\cdot\int_0^1 (f''(x))^2dx$$

and $\int_0^1 (1-x)f''(x)dx=(1-x)f'(x)\Big|_0^1+\int_0^1 f'(x)dx=-a$

Hence $\int_0^1 (f''(x))^2dx \ge 3a^2$ with equality attained at

$f''(x) = \lambda(1 - x), \lambda \in R$

9. (a) : Given integral

$$=\int\frac{(a\cos x+c)}{(1-e\cos x)^2}dx+\int\frac{b\sin x}{(1-e\cos x)^2}dx=I_1+I_2(\text{say})$$

I_2 is rational function of (sin$x$, cos$x$).

For $I_1=\int\frac{(a\cos x+c)}{(1-e\cos x)^2}dx$ [Dividing by $\sin^2 x$ both numerator and denominator]

$$I_1=\int\frac{(a\cot x\ \text{cosec}x+c\ \text{cosec}^2x)dx}{(\text{cosec}x-e\cot x)^2}$$

Now, we take (cosec$x$ – $e$cot$x$) = t

then (–cosec$x$ cot$x$ + $e$cosec^2x)dx = dt

So, I_1 is a rational function of (sin$x$, cos$x$) if we have

$$\frac{a}{-1}=\frac{c}{e}\ \ i.e.,\ ae+c=0$$

10. (c) : $f(x)$ has primitive function only when it is continuous in $\left[\frac{-1}{2},\frac{1}{2}\right]$, $i.e., \lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^-} f(x)$

Hence, $k=\frac{-\pi}{2}$

11. (b) : $I=\int\frac{2\sin x+1}{(\sin x+2)^2}dx=\int\frac{2\tan x\sec x+\sec^2 x}{(\tan x+2\sec x)^2}dx$

Now, take tan$x$ + 2sec$x$ = t

12. (d) : Notice that $1 + x(x + 1)(x + 2)(x + 3)$

$= 1 + (x^2 + 3x)(x^2 + 3x + 2) = 1 + t(t + 2)$

where $t = x^2 + 3x$

$= 1 + t^2 + 2t = (t + 1)^2 = (x^2 + 3x + 1)^2$

So, $I=\int\frac{(2x+3)dx}{(x^2+3x+1)^2}$. Take $x^2+3x+1=u$

13. (c) : Convert tan and cot into sin and cos and take resulting denominator

$\sin^{21}x + \cos^{21}x$ as t we get $k = 21$

14. (a) : Let $f^{-1}(x) = y \Rightarrow x = f(y)$ and $dx = f'(y)dy$

$$\text{So, } I_2=\int_a^b f(y)(b-y)f'(y)dy=\int_a^b (b-y)\cdot(f(y)f'(y))dy$$

$$=\frac{1}{2}\cdot\int_a^b [f^2(y)-f^2(a)]dy\ \ [\text{By parts, }(b-y)\text{ as }1^{\text{st}}\text{ function}]$$

$$\Rightarrow \quad 2I_2=\int_a^b [f^2(x)-f^2(a)]dx=I_1$$

15. (a) : Given integral

$$=\int e^{x^2}dx+\int 2x^2e^{x^2}dx=I_1+I_2(\text{say})$$

For, $I_2=\int x\cdot(2xe^{x^2})dx$

$=x\cdot e^{x^2}-\int e^{x^2}dx=xe^{x^2}-I_1$ [Integrating by parts]

Hence, $I_1+I_2=xe^{x^2}+c$

16. (a) : Putting $x^3 + 1 = t^2$, we have

$$f(x)=\sqrt{t^2+1-2t}+\sqrt{t^2+9-6t}=|t-1|+|t-3|$$

For $x \in [0, 2]$, we have $t \in [1, 3]$ Hence,

$f(x) = (t - 1) + (3 - t) = 2$ $i.e., \int f(x)dx=2x+c$

17. (d) : Putting $t = 1 - x$ gives

$$I=\int_0^1 \sqrt[3]{2(1-t)^3-3(1-t)^2-(1-t)+1}\,dt$$

$= -I$ on simplification

Hence, $2I = 0$ $i.e., I = 0$

18. (b) : The given integral equation can be rewritten as,

$$\int_0^1 2xf(x^2)dx=\int_0^1 x^2dx+\int_0^1 f^2(x^2)dx$$

$$\text{or, } \int_0^1 [f^2(x^2)+x^2-2xf(x^2)]dx=0$$

$$i.e., \int_0^1 (f(x^2)-x)^2dx=0$$

$\Rightarrow$ $f(x^2) - x = 0$ for all $x \in [0, 1]$

Hence, $f(x)=\sqrt{x}$ $i.e., f\left(\frac{1}{2}\right)=\frac{1}{\sqrt{2}}$.

JEE Main 2018

MOCK TEST PAPER

Series-2

Time: 1 hr 15 min.

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

Unit No.	Topic	Syllabus In Details
UNIT NO.2	Complex Numbers	Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, square root of a complex number, cube roots of unity, geometric interpretations.
	Quadratic Equations	Quadratic equations in real and complex system and their solutions. Quadratic equations with real coefficients, relation between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.
	Trigonometry	Compound angles, Transformation of sum & product, Multiple & sub-multiple angles.
	Co-ordinate Geometry-2D	Straight lines: Equation of a straight line in various forms, angle between two lines, distance of a point from a line, lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines, centroid, orthocenter, incentre and circumcentre of a triangle.

1. If ω is a cube root of unity, then

$$\sin\left\{(\omega^{35}+\omega^{25})\pi+\frac{\pi}{2}\right\}+\cos\left\{(\omega^{10}+\omega^{23})\pi-\frac{\pi}{4}\right\}$$

is

(a) $\dfrac{2+\sqrt{2}}{2}$ (b) $\dfrac{2+\sqrt{2}}{\sqrt{2}}$

(c) $-\left(\dfrac{2+\sqrt{2}}{2}\right)$ (d) $\dfrac{2-\sqrt{2}}{2}$

2. $\sin^{-1}\left[\frac{1}{i}(z-1)\right]$, where z is non-real, can be the angle of a triangle if

(a) $\text{Re}(z) = 1, \text{Im}(z) = 2$

(b) $\text{Re}(z) = 1, -1 \le \text{Im}(z) \le 1$

(c) $\text{Re}(z) + \text{Im}(z) = 0$

(d) $\text{Re}(z) = \text{Im}(z)$

3. Origin and the roots of $z^2 + pz + q = 0$ form an equilateral triangle if

(a) $p^2 = q$ (b) $p^2 = 3q$

(c) $q^2 = 3p$ (d) $q^2 = p$

4. If $\dfrac{5z_2}{11z_1}$ is purely imaginary, then $\left|\dfrac{2z_1+3z_2}{2z_1-3z_2}\right|$ is

(a) $\dfrac{37}{33}$ (b) $\dfrac{11}{5}$ (c) 1 (d) $\dfrac{5}{11}$

5. Point at which the curves $\arg(z-3i)=\dfrac{3\pi}{4}$ and $\arg(2z+1-2i)=\dfrac{\pi}{4}$ intersect, is

(a) $\left(\dfrac{3}{4},\dfrac{9}{4}\right)$ (b) $\left(\dfrac{3}{4},\dfrac{7}{4}\right)$

(c) $\left(\dfrac{2}{4},\dfrac{7}{4}\right)$ (d) $\left(\dfrac{6}{7},\dfrac{4}{7}\right)$

6. Square root(s) of $-i$ is/are

(a) $\dfrac{1}{\sqrt{2}}(1-i)$ (b) $\dfrac{1}{\sqrt{3}}(i-1)$

(c) $\pm\dfrac{1}{2}(1-i)$ (d) $-\dfrac{1}{\sqrt{2}}(1-i)$

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397.

7. The equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 3 = 0$ have one root in common, then
(a) $a : b : c = 1 : 2 : 3$ (b) $a : b : c = 3 : 2 : 1$
(c) $a : b : c = 1 : 3 : 2$ (d) none of these.

8. If 1 lies between the roots of the equation $y^2 - my + 1 = 0$ and $[x]$ denotes greatest integer $\leq x$ then $\left[\left(\frac{4|x|}{|x|^2+16}\right)^m\right]$ is equal to
(a) 0 (b) 1
(c) 2 (d) none of these.

9. If the inequality $\frac{mx^2+3x+4}{x^2+2x+2} < 5$ is satisfied for all $x \in R$ then
(a) $1 < m < 5$ (b) $-1 < m < 5$
(c) $1 < m < 6$ (d) $m < \frac{71}{24}$

10. A quadratic equation whose roots are $\left(\frac{\gamma}{\alpha}\right)^2$ and $\left(\frac{\beta}{\alpha}\right)^2$, where α, β, γ are roots of $x^3 + 27 = 0$ is
(a) $x^2 - x + 1 = 0$ (b) $x^2 + 3x + 9 = 0$
(c) $x^2 + x + 1 = 0$ (d) $x^2 - 3x - 9 = 0$

11. If the mapping $f(x) = ax + b$, $a < 0$ maps $[-1, 1]$ on to $[0, 2]$, then for all values of θ, $A = \cos^2\theta + \sin^4\theta$ is such that
(a) $f\left(\frac{1}{4}\right) \leq A \leq f(0)$ (b) $f(0) \leq A \leq f(-2)$
(c) $f\left(\frac{1}{3}\right) \leq A \leq f(0)$ (d) $f(-1) \leq A \leq f(-2)$

12. If α and β are the solutions of $\sin^2 x + a\sin x + b = 0$ as well as that of $\cos^2 x + c\cos x + d = 0$, then $\sin(\alpha + \beta)$ is equal to
(a) $\frac{2bd}{b^2+d^2}$ (b) $\frac{a^2+c^2}{2ac}$
(c) $\frac{b^2+d^2}{2bd}$ (d) $\frac{2ac}{a^2+c^2}$

13. The minimum value of $2^{\sin x} + 2^{\cos x}$ is
(a) 1 (b) $2-\frac{1}{\sqrt{2}}$
(c) $2^{-\frac{1}{\sqrt{2}}}$ (d) $2^{1-\frac{1}{\sqrt{2}}}$

14. If $a\sin x + b\cos(x + \theta) + b\cos(x - \theta) = d$ for some real x, then the minimum value of $|\cos\theta|$ is equal to
(a) $\frac{1}{2|b|}\sqrt{d^2-a^2}$ (b) $\frac{1}{2|a|}\sqrt{d^2-a^2}$
(c) $\frac{1}{|d|}\sqrt{d^2-a^2}$ (d) none of these.

15. If $\cos 25° + \sin 25° = p$, then $\cos 50° =$
(a) $\sqrt{2-p^2}$ (b) $-p\sqrt{2-p^2}$
(c) $p\sqrt{2-p^2}$ (d) $-\sqrt{2-p^2}$

16. $\cos^2 A + \cos^2(B - A) - 2\cos A \cos B \cos(A - B) =$
(a) $\cos 2A$ (b) $\sin^2 A$ (c) $\sin^2 B$ (d) $\cos^2 B$

17. If $\tan\alpha = \frac{x^2-x}{x^2-x+1}$ and $\tan\beta = \frac{1}{2x^2-2x+1}$, $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha + \beta =$
(a) $\pi/4$ (b) $\pi/2$ (c) $\pi/3$ (d) $3\pi/4$

18. In a triangle one angle is $3\pi/4$ and other two angles are two values of θ satisfying $a\tan\theta + b\sec\theta = c$ where $|b| \leq \sqrt{a^2+c^2}$. Then $a^2 - c^2$ is equal to
(a) ac (b) $2ac$ (c) a/c (d) $2a/c$

19. If $\cos(x - y)$, $\cos x$, $\cos(x + y)$ are in H.P. then $\left|\cos x \sec\frac{y}{2}\right|$ equals
(a) 1 (b) 2 (c) $\sqrt{2}$ (d) 3

20. In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the equation $\log_{\sin\theta}(\cos 2\theta) = 2$ has
(a) no solution (b) a unique solution
(c) two solutions
(d) infinitely many solutions.

21. Area of the parallelogram formed by the set of parallel lines
$2x + 3y - 4 = 0$, $2x + 3y - 7 = 0$ and
$3x - 4y + 1 = 0$, $3x + 4y + 7 = 0$ is
(a) 18 sq. units (b) 15 sq. units
(c) 13 sq. units (d) none of these

22. If $A(0, 6)$, $B(2, 1)$, $C(7, 3)$ are vertices of a square $ABCD$, then equation of diagonal BD =
(a) $x + y - 3 = 0$ (b) $3x - y - 5 = 0$
(c) $2x + 3y - 7 = 0$ (d) none of these

23. A straight line through the origin meet the parallel lines $2x + 3y = 6$ and $4x + 6y = -9$ at points A and B respectively. Then the point(origin) divides the segment AB in the ratio

(a) 3 : 4 (b) 2 : 3
(c) 4 : 3 (d) 3 : 2

24. If A is (2, 0) and B is (0, 2) then the coordinate of the point P on the line $2x + 3y + 1 = 0$ such that $|PA - PB|$ is maximum.
(a) (7, 5) (b) (7, –5) (c) (5, 7) (d) (5, –7)

25. The range of values of θ, $\theta \in [0, 2\pi]$ for which (cosθ, sinθ) lies inside the triangle formed by $x + y - 2 = 0$, $6x+2y-\sqrt{10}=0$ and $x - y - 1 = 0$

(a) $0<\theta<\frac{3\pi}{2}$

(b) $\theta<\frac{5\pi}{6}-\tan^{-1}3$

(c) $0<\theta<\frac{5\pi}{6}-\tan^{-1}3$

(d) none of these.

26. The slopes of three sides of a triangle are –1, –2, 3. If the orthocentre of the triangle is the origin, then locus of the centroid of the triangle is
(a) $2x + 9y = 0$ (b) $2x - 9y = 0$
(c) $9x - 2y = 0$ (d) $9x + 2y = 0$

27. The straight line $ax + by + c = 0$ cuts the locus of point of intersection of the lines $\frac{tx}{4}-\frac{y}{3}+t=0$, $\frac{x}{4}+\frac{ty}{3}-1=0$ at A and B such that the line AB subtends a right angle at the origin. The distance of the given line from the origin is

(a) $\frac{12}{5}$ (b) $\frac{5}{12}$ (c) $\frac{25}{144}$ (d) $\frac{144}{25}$

28. The line $3x - 4y + 8 = 0$ is rotated through an angle $\pi/4$ in the clock wise direction about the point (0, 2). The equation of the line in its new position is
(a) $7y + x - 14 = 0$ (b) $7y - x - 14 = 0$
(c) $7y + x - 2 = 0$ (d) $7y - x = 0$

29. If the straight lines $2x + 3y - 1 = 0$ and $ax + by - 1 = 0$ form a triangle with origin as orthocentre, then (a, b) is given by
(a) (6, 4) (b) (–3, 3)
(c) (–8, 8) (d) (0, 7)

30. If the lines $x = a + m$, $y = -2$ and $y = mx$ are concurrent, the least value of $|a|$ is
(a) 0 (b) $\sqrt{2}$
(c) $2\sqrt{2}$ (d) none of these.

SOLUTIONS

1. (c) : $\sin\left\{(\omega^{35}+\omega^{25})\pi+\frac{\pi}{2}\right\}+\cos\left\{(\omega^{10}+\omega^{23})\pi-\frac{\pi}{4}\right\}$

$=\sin\left\{(\omega^2+\omega)\pi+\frac{\pi}{2}\right\}+\cos\left\{(\omega+\omega^2)\pi-\frac{\pi}{4}\right\}$ $[\because\ \omega^3=1]$

$=\sin\left(-\pi+\frac{\pi}{2}\right)+\cos\left(-\pi-\frac{\pi}{4}\right)$ $[\because\ 1+\omega+\omega^2=0]$

$=\sin\left(-\frac{\pi}{2}\right)+\cos\left(\pi+\frac{\pi}{4}\right)=-1-\frac{1}{\sqrt{2}}=-\left(\frac{2+\sqrt{2}}{2}\right)$

2. (b) : $\sin^{-1}\left(\frac{z-1}{i}\right)$ will be an angle of a triangle if $\frac{z-1}{i}$ is real.

Let $z = x + iy$, therefore

$\frac{z-1}{i}=\frac{x+iy-1}{i}=\frac{i(x+iy-1)}{-1}=y-i(x-1)$ is real

$\Rightarrow$ $x - 1 = 0$ *i.e.* $x = 1$

$\therefore\ \sin^{-1}\frac{z-1}{i}=\sin^{-1}y\ \therefore\ -1\le y\le 1$

3. (b) : Let z_1, z_2 are the roots of the equation $z^2 + pz + q = 0$

$\therefore$ $z_1 + z_2 = -p$ and $z_1z_2 = q$

Now, since origin, z_1 and z_2 are vertices of an equilateral triangle $\therefore$ $\frac{z_2}{z_1}=e^{i\frac{\pi}{3}}$

$\Rightarrow\ \frac{z_2}{z_1}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

Now $\frac{z_2}{z_1}+1=\frac{3}{2}+i\frac{\sqrt{3}}{2}$ $\Rightarrow$ $\frac{z_1+z_2}{z_1}=\frac{3}{2}+i\frac{\sqrt{3}}{2}$

$\Rightarrow\ -p=z_1\sqrt{3}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$

$\Rightarrow\ p^2=z_1^2 3e^{i\frac{\pi}{3}}=3z_1\left|z_1e^{i\frac{\pi}{3}}\right|=3z_1z_2=3q$

4. (c) : Given that $\frac{5z_2}{11z_1}$ is purely imaginary

$\Rightarrow\ \frac{z_2}{z_1}=\frac{11}{5}ik$

$$\therefore\ \left|\frac{2z_1+3z_2}{2z_1-3z_2}\right| = \left|\frac{1+\frac{3}{2}\frac{z_2}{z_1}}{1-\frac{3}{2}\frac{z_2}{z_1}}\right| = \left|\frac{1+\frac{33}{10}ik}{1-\frac{33}{10}ik}\right| = 1$$

$$\left[\because\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } |z| = |\bar{z}|\right]$$

5. (a) : Let $z = x + iy$

$\therefore\ z-3i = x+iy-3i = x+i(y-3)$

So, $\arg(x + i(y - 3)) = \frac{3\pi}{4}$

$\Rightarrow\ \tan^{-1}\left(\frac{y-3}{x}\right) = \frac{3\pi}{4}\ \Rightarrow\ x + y = 3$ ...(i)

Again, $2z+1-2i = 2(x+iy)+1-2i = 2x+1+i(2y-2)$

$\therefore\ \arg(2z+1-2i) = \frac{\pi}{4} \Rightarrow \arg\{(2x+1)+i(2y-2)\} = \frac{\pi}{4}$

$\Rightarrow\ \tan^{-1}\left(\frac{2y-2}{2x+1}\right) = \frac{\pi}{4}\ \Rightarrow\ \frac{2y-2}{2x+1} = 1$

$\Rightarrow\ 2x - 2y = -3$ ...(ii)

Solving (i) and (ii) we get, $x = \frac{3}{4}, y = \frac{9}{4}$

6. (a) : $-i = \frac{1}{2}(-2i) = \frac{1}{2}(1^2+i^2-2i) = \frac{1}{2}(1-i)^2$

$\therefore\ \sqrt{-i} = \frac{1}{\sqrt{2}}(1-i)$

7. (a) : The equation $x^2 + 2x + 3 = 0$ has imaginary roots, therefore both the roots of the given equations will be common. $\therefore\ \frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

8. (a) : Since 1 lies between the roots of $y^2 - my + 1 = 0$

Let $f(y) = y^2 - my + 1$

$\therefore\ f(1) < 0\ \Rightarrow\ 2 - m < 0\ \Rightarrow\ m > 2$

Now A.M ≥ G.M $\Rightarrow\ \frac{|x|^2+16}{2} \ge 4|x|$

$\therefore\ \frac{1}{2} \ge \frac{4|x|}{|x|^2+16}\ \Rightarrow\ 0 \le \frac{4|x|}{|x|^2+16} \le \frac{1}{2}$

$\Rightarrow\ 0 \le \left(\frac{4|x|}{|x|^2+16}\right)^m < 1\ \therefore\ \left[\frac{4|x|}{|x|^2+16}\right] = 0$

9. (d) : We have $\frac{mx^2+3x+4}{x^2+2x+2} < 5$

$\Rightarrow\ mx^2 + 3x + 4 < 5(x^2 + 2x + 2)$

$\Rightarrow\ (5 - m)x^2 + 7x + 6 > 0$

i.e. $49 + 24(m - 5) < 0$ and $(5 - m) > 0$

$\Rightarrow\ m < \frac{71}{24}$ and $m < 5\ \Rightarrow\ m < \frac{71}{24}$

10. (c) : Given that α, β, γ are the roots of $x^3 + 27 = 0$

$\Rightarrow\ (x + 3)(x^2 - 3x + 9) = 0$

$\therefore\ x = -3, -3\omega, -3\omega^2$, where ω is an imaginary cube roots of unity. $\therefore\ \alpha = -3, \beta = -3\omega, \gamma = -3\omega^2$

so, $\left(\frac{\gamma}{\alpha}\right)^2 = (\omega^2)^2 = \omega$ and $\left(\frac{\beta}{\alpha}\right)^2 = (\omega)^2 = \omega^2$

Thus, the equation whose roots are $\left(\frac{\gamma}{\alpha}\right)^2$ and $\left(\frac{\beta}{\alpha}\right)^2$ is $x^2-(\omega+\omega^2)x+\omega^3 = 0\ \Rightarrow\ x^2+x+1 = 0$

11. (a) : Given that $f(x) = ax + b\ \therefore f'(x) = a$

Since $a < 0$, $f(x)$ is a decreasing function

$\therefore f(-1) = 2$ and $f(1) = 0 \Rightarrow -a + b = 2$ and $a + b = 0$

$\therefore\ a = -1$ and $b = 1$. Thus $f(x) = -x + 1$

Clearly $f(0) = 1$,

$f\left(\frac{1}{4}\right) = \frac{3}{4}, f(-2) = 3, f\left(\frac{1}{3}\right) = \frac{2}{3}, f(-1) = 2$

Also $A = \cos^2\theta + \sin^4\theta = \frac{1+\cos 2\theta}{2} + \left(\frac{1-\cos 2\theta}{2}\right)^2$

$= \frac{1}{2} + \frac{1}{2}\cos 2\theta + \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta$

$= \frac{3}{4} + \frac{1}{4}\left(\frac{1+\cos 2\theta}{2}\right) = \frac{7}{8} + \frac{1}{8}\cos 4\theta$

$\therefore\ \frac{3}{4} \le A \le 1 \Rightarrow f\left(\frac{1}{4}\right) \le A \le f(0)$

12. (d) : From the given condition,

we have $\sin\alpha + \sin\beta = -a$ and $\cos\alpha + \cos\beta = -c$

$\Rightarrow\ 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -a$

and $2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -c\ \Rightarrow\ \tan\frac{\alpha+\beta}{2} = \frac{a}{c}$

$\Rightarrow\ \sin(\alpha+\beta) = \frac{2\tan\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}} = \frac{2ac}{a^2+c^2}$

13. (d) : $2^{\sin x} + 2^{\cos x} \ge 2\cdot\sqrt{2^{\sin x}\cdot 2^{\cos x}}$

[since A.M ≥ G.M]

$\geq 2\sqrt{2^{\sin x+\cos x}} \geq 2\sqrt{2^{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)}}$

$\geq 2\sqrt{2^{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}} \geq 2\sqrt{2^{-\sqrt{2}}} \geq 2^{1-\frac{1}{\sqrt{2}}}$

14. (a) : $a\sin x + b\cos x(x+\theta) + b\cos(x-\theta) = d$

$\Rightarrow\ a\sin x + b\{\cos(x+\theta) + \cos(x-\theta)\} = d$

$\Rightarrow\ a\sin x + 2b\cos x\cos\theta = d \Rightarrow \sqrt{a^2+4b^2\cos^2\theta}\ \times$

$\left(\frac{a}{\sqrt{a^2+4b^2\cos^2\theta}}\sin x+\frac{2b\cos\theta}{\sqrt{a^2+4b^2\cos^2\theta}}\cos x\right)=d$

$\Rightarrow \sqrt{a^2+4b^2\cos^2\theta}\sin(x+\alpha)=d,$

where $\cos\alpha=\frac{a}{\sqrt{a^2+4b^2\cos^2\theta}}$

$\Rightarrow \sin(x+\alpha)=\frac{d}{\sqrt{a^2+4b^2\cos^2\theta}}$

$\Rightarrow d\leq\sqrt{a^2+4b^2\cos^2\theta}\quad [\because \sin\theta\leq 1]$

$\Rightarrow \cos^2\theta\geq\frac{d^2-a^2}{4b^2} \Rightarrow |\cos\theta|\geq\frac{\sqrt{d^2-a^2}}{2|b|}$

15. (c) : Given, $\cos 25° + \sin 25° = p$

Now, $\cos 50° = \cos^2 25 - \sin^2 25°$

$= (\cos 25° + \sin 25°)(\cos 25° - \sin 25°)$

$= p(\cos 25° - \sin 25°)$ ...(i)

But $(\cos 25° + \sin 25°)^2 + (\cos 25° - \sin 25°)^2 = 2$

$\Rightarrow (\cos 25°-\sin 25°)=\sqrt{2-p^2}\quad [\because \cos 25°-\sin 25°>0]$

Now (i) becomes $\cos 50°=p\sqrt{2-p^2}$

16. (c) : $\cos^2 A + \cos^2(B-A) - 2\cos A\cos B\cos(A-B)$

$= \cos^2 A + \cos^2(B-A) - \cos(A-B)$
$\{\cos(A+B) + \cos(A-B)\}$

$= \cos^2 A + \cos^2(B-A) - \{\cos^2 A - \sin^2 B\} - \cos^2(A-B)$

$= \sin^2 B$

17. (a) : Given that $\tan\alpha=\frac{x^2-x}{x^2-x+1}$

and $\tan\beta=\frac{1}{2x^2-2x+1}, 0<\alpha,\beta<\frac{\pi}{2}$

Let $x^2-x=m \ \therefore\ \tan\alpha=\frac{m}{m+1}$ and $\tan\beta=\frac{1}{2m+1}$

Now $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$=\frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{(m+1)(2m+1)}}=\frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\left(\frac{1}{m+1}-\frac{1}{2m+1}\right)}$

$=\frac{\frac{m}{m+1}+\frac{1}{2m+1}}{\frac{m}{m+1}+\frac{1}{2m+1}}=1$

Hence, $\alpha+\beta=\frac{\pi}{4}\ \left(\text{As } 0<\alpha,\beta<\frac{\pi}{2}\right)$

18. (b) : Given that, $a\tan\theta + b\sec\theta = c$

$\Rightarrow c\cos\theta - a\sin\theta = b$

Let α and β be the other two angles of the triangle

$\Rightarrow\ c\cos\alpha - a\sin\alpha = c\cos\beta - a\sin\beta$

$\Rightarrow\ c(\cos\alpha - \cos\beta) = a(\sin\alpha - \sin\beta)$

$\Rightarrow 2c\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}=2a\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\Rightarrow \tan\frac{\alpha+\beta}{2}=-\frac{a}{c}$

Now $\tan(\alpha+\beta)=\frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1-\tan^2\frac{\alpha+\beta}{2}}$

$\Rightarrow\ 1=\frac{2\left(-\frac{a}{c}\right)}{1-\frac{a^2}{c^2}} \Rightarrow a^2-c^2=2ac$

19. (c) : Given, $\cos(x-y)$, $\cos x$, $\cos(x+y)$ are in H.P.

$\therefore\ \frac{2}{\cos x}=\frac{1}{\cos(x-y)}+\frac{1}{\cos(x+y)}$

$=\frac{\cos(x+y)+\cos(x-y)}{\cos(x+y)\cos(x-y)}=\frac{2\cos x\cos y}{\cos(x+y)\cos(x-y)}$

$\Rightarrow\ \cos x=\frac{\cos(x+y)\cos(x-y)}{\cos x\cos y}$

$\Rightarrow\ \cos^2 x(\cos y) = \cos^2 x - \sin^2 y$

$\Rightarrow\ \cos^2 x(\cos y - 1) = -\sin^2 y = -(1-\cos y)(1+\cos y)$

$\Rightarrow\ \cos^2 x=1+\cos y=2\cos^2\frac{y}{2}\Rightarrow\left|\cos x\sec\frac{y}{2}\right|=\sqrt{2}$

20. (b) : $\log_{\sin\theta}(\cos 2\theta)=2\Rightarrow\sin^2\theta=\cos 2\theta$

$=1-2\sin^2\theta\Rightarrow 3\sin^2\theta=1\Rightarrow\sin^2\theta=\frac{1}{3}\ \therefore\ \sin\theta=\frac{1}{\sqrt{3}}$

$\left[\because -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \;\therefore\; -1 \le \sin\theta \le 1 \text{ here, } 0 < \sin\theta < 1\right]$

a unique solution.

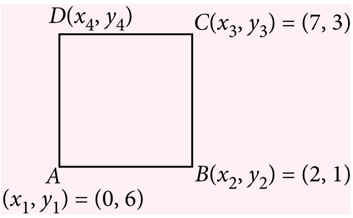
21. (a) : Area of a parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ is given by

$$\text{Area of parallelogram} = \left|\frac{(c_1 - d_1)\cdot(c_2 - d_2)}{a_1b_2 - b_1a_2}\right|$$

$$= \left|\frac{(-4+7)(1-7)}{8-9}\right| = 18$$

$\therefore$ Required area = 18 square units

22. (d) :



Here, $\frac{3-y_4}{7-x_4} = \frac{-5}{2} \;\Rightarrow\; 5x_4 + 2y_4 = 41$ ... (i)

Also, $\frac{6-y_4}{0-x_4} = \frac{1-3}{2-7} = \frac{2}{5} \;\Rightarrow\; 2x_4 - 5y_4 = -30$ ... (ii)

Solving (i) and (ii), we get $x = 5$ and $y = 8$

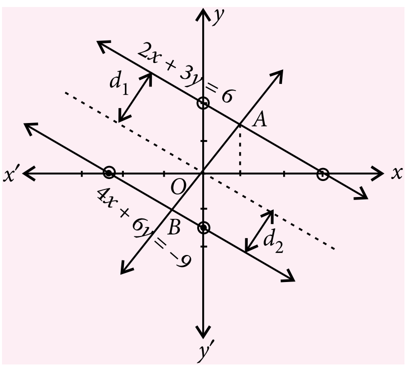
Now slope of $BD = \frac{8-1}{5-2} = \frac{7}{3}$

$\therefore$ Equation of BD is $y - 8 = \frac{7}{3}(x-5) \Rightarrow 7x - 3y = 11$

23. (c) : The given equations of the straight line are

$2x + 3y = 6 \;\Rightarrow\; \frac{x}{3} + \frac{y}{2} = 1$ ... (i)

and $4x + 6y = -9 \;\Rightarrow\; \frac{x}{-9/4} + \frac{y}{-3/2} = 1$ ... (ii)



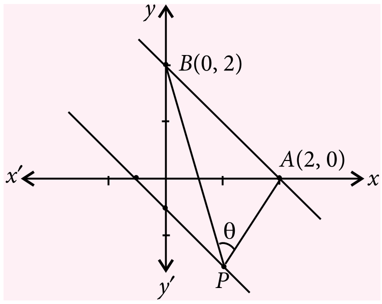
From the above diagram, we have

$$\frac{AO}{OB} = \frac{d_1}{d_2} = \frac{\text{Distance of orgin from } 2x + 3y = 6}{\text{Distance of orgin from } 4x + 6y = -9}$$

$$= \frac{12}{9} = \frac{4}{3}$$

$\therefore$ The required ratio is 4 : 3.

24. (b) :



Now applying cosine rule in ΔAPB,

$AB^2 = AP^2 + PB^2 - 2PA\cdot PB\cos\theta$

$\Rightarrow AB^2 \ge AP^2 + PB^2 - 2PA\cdot PB$

$\Rightarrow AB^2 \ge (PA - PB)^2 \Rightarrow |PA - PB| \le AB$

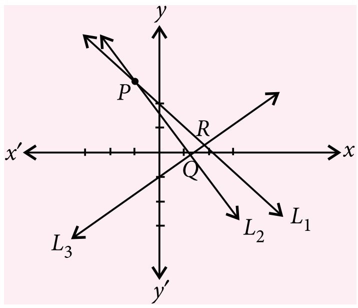
$\Rightarrow |PA - PB| \le \sqrt{8} \quad [\because AB = \sqrt{8}]$

$\Rightarrow$ Maximum value of $|PA - PB| = 2\sqrt{2}$ for $\theta = 0$

$\theta = 0$ means that the point of intersection of $x + y = 2$ (line AB) and $2x + 3y + 1 = 0$ is the required point $P \equiv (7, -5)$.

25. (c) : Let $L_1 \equiv x + y - 2 = 0$,

$L_2 = 6x + 2y - \sqrt{10} = 0$ and $L_3 = x - y - 1 = 0$



Let $(\cos\theta, \sin\theta)$ lies inside ΔPQR,

$\cos\theta + \sin\theta - 2 < 0$ ... (i)

$\cos\theta - \sin\theta - 1 < 0$ ... (ii)

and $6\cos\theta + 2\sin\theta - \sqrt{10} > 0$ ... (iii)

From (i), we have $\cos\theta + \sin\theta < 2$ if true $\forall\ \theta$

From (ii) we have, $\cos\theta - \sin\theta < 1 \;\Rightarrow\; \sin\left(\frac{\pi}{4} - \theta\right) < \frac{1}{\sqrt{2}}$

$\Rightarrow\; 0 < \theta < \frac{3\pi}{2} \quad (\because\ \theta \in [0, 2\pi])$ ...(*)

From (iii), we have $6\cos\theta + \sin\theta > \sqrt{10}$

$$\Rightarrow \sin(\theta+\alpha) > \frac{1}{2} \text{ where } \tan\alpha = 3$$

$$\Rightarrow \frac{\pi}{6} < (\theta+\alpha) < \frac{5\pi}{6} \Rightarrow \theta+\alpha < \frac{5\pi}{6}$$

$$\Rightarrow \theta < \frac{5\pi}{6} - \alpha \quad \left(\because \theta+\alpha > \frac{\pi}{6} \forall \theta, \alpha > \frac{\pi}{6}\right)$$

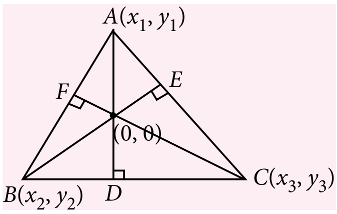
$$\Rightarrow \theta < \frac{5\pi}{6} - \tan^{-1}3 \qquad \dots (**)$$

Combining (*) and (**), we get $0 < \theta < \frac{5\pi}{6} - \tan^{-1}3$

26. (b) : Given that slope of $BC = -1$, $AC = -2$ and $AB = 3$

$\therefore$ Slope of $AD = 1$

$\Rightarrow y_1 = x_1$



Slope of $BE = \frac{1}{2} \Rightarrow y_2 = \frac{1}{2}x_2 \Rightarrow x_2 = 2y_2$

Slope of $CF = -\frac{1}{3} \Rightarrow y_3 = -\frac{1}{3}x_3 \Rightarrow x_3 = -3y_3$

Also, $y_2 - y_3 = -1(x_2 - x_3)$

$y_3 - y_1 = -2(x_3 - x_1)$

$y_1 - y_2 = 3(x_1 - x_2)$

Solving for x_2, x_3, y_1, y_2, y_3 in terms of x_1, we get

$$x_2 = \frac{4x_1}{5},\ x_3 = \frac{9x_1}{5},\ y_1 = x_1,\ y_2 = \frac{2x_1}{5},\ y_3 = -\frac{3x_1}{5}$$

Now, centroid $= (h, k) \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$$\Rightarrow 3h = x_1 + \frac{4x_1}{5} + \frac{9x_1}{5} \Rightarrow 3h = \frac{18x_1}{5} \qquad \dots\text{(i)}$$

$$\text{and } 3k = x_1 + \frac{2x_1}{5} - \frac{3x_1}{5} \Rightarrow 3k = \frac{4x_1}{5} \qquad \dots\text{(ii)}$$

From (i) & (ii), we get $2h = 9k$

$\therefore$ Locus of (h, k) is $2x = 9y$

27. (a) : The given lines are

$$\frac{tx}{4} - \frac{y}{3} + t = 0 \quad \text{and} \quad \frac{x}{4} + \frac{ty}{3} - 1 = 0$$

Eliminating t, we get $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Homogenising $\frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$ with $ax + by + c = 0$, we get

$$\frac{x^2}{16} + \frac{y^2}{9} - 1\left(\frac{ax+by}{-c}\right)^2 = 0$$

Coefficient of x^2 + Coefficient of y^2 = 0

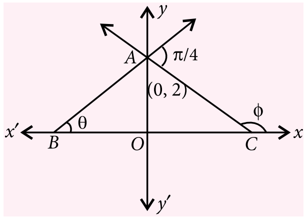
$$\Rightarrow \frac{1}{16} - \frac{a^2}{c^2} + \frac{1}{9} - \frac{b^2}{c^2} = 0 \Rightarrow \frac{25}{144} = \frac{a^2+b^2}{c^2}$$

$$\Rightarrow \frac{c^2}{a^2+b^2} = \frac{144}{25} \Rightarrow \frac{|c|}{\sqrt{a^2+b^2}} = \frac{12}{5}$$

28. (a) : The equation of the given line is

$3x - 4y + 8 = 8$ ...(i)

$$\Rightarrow \frac{x}{-8/3} + \frac{y}{2} = 1$$



The slope of line (i) is $\frac{3}{4}$

$$\therefore \tan\theta = \frac{3}{4};\ \phi = \theta + \frac{3\pi}{4}$$

$$m = \tan\phi = \tan\left(\theta + \frac{3\pi}{4}\right) = \frac{\tan\theta - 1}{1+\tan\theta} = \frac{\frac{3}{4}-1}{1+\frac{3}{4}} = -\frac{1}{7}$$

Equation of AC is $y - 2 = -\frac{1}{7}(x-0)$ *i.e.* $x + 7y - 14 = 0$

29. (c) : Equation of AO is

$(2x + 3y - 1) + \lambda(x + 2y - 1) = 0$ ...(i)

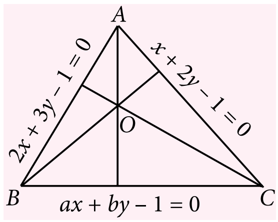
Since (i) is passing through (0, 0).

$\therefore \lambda = -1$, (i) becomes $x + y = 0$

Again $AO \perp BC$

$$\therefore (-1)\left(-\frac{a}{b}\right) = -1$$

$\Rightarrow a = -b$



Similarly, $(2x + 3y - 1) + \mu(ax - ay - 1) = 0$ will be the equation of BO for $\mu = -1$

$$BO \perp AC \Rightarrow \left\{-\frac{(2-a)}{(3+a)}\right\}\left(-\frac{1}{2}\right) = -1$$

$\therefore a = -8, b = 8$

30. (c) : The equation of the given lines are

$\lambda = a + m,\ y = -2$ and $y = mx$

Since the lines are concurrent

$$\therefore \begin{vmatrix} 1 & 0 & -a-m \\ 0 & 1 & 2 \\ m & -1 & 0 \end{vmatrix} = 0$$

$\Rightarrow 1(0 + 2) + 0 - (a + m)(-m) = 0$

$\Rightarrow m^2 + am + 2 = 0$

$\because m$ is real, $\therefore a^2 - 4 \cdot 1 \cdot 2 \geq 0$

$\Rightarrow a^2 \geq 8 \Rightarrow |a| \geq 2\sqrt{2}$

Hence, the least value of $|a|$ is $2\sqrt{2}$.

MATHS MUSING

SOLUTION SET-175

1. (d) : The image of z in the real axis is $\bar{z}$ and $\arg\bar{z} = -\arg z = \arg\frac{1}{z}$. The desired image is given by $\arg\left(\frac{\bar{z}-3}{\bar{z}-i}\right) = \frac{\pi}{6}$ $\therefore$ $\arg\left(\frac{\bar{z}-3}{\bar{z}-i}\right) = -\frac{\pi}{6}$

$$\Rightarrow \arg\left(\frac{z-3}{z+i}\right) = -\frac{\pi}{6} \Rightarrow \arg\left(\frac{z+i}{z-3}\right) = \frac{\pi}{6}$$

2. (c) : $AI + AB = BC \Rightarrow \dfrac{r}{\sin\frac{A}{2}} = a - c$

$$\Rightarrow 4R\sin\frac{B}{2}\sin\frac{C}{2} = 2R(\sin A - \sin C)$$

$$\therefore \sin\frac{C}{2} = \sin\frac{A-C}{2} \Rightarrow A = 2C$$

$B = \pi - (A + C) = \pi - 3C \Rightarrow B + 3C = \pi$

3. (b) : $\dfrac{dy}{dx} = xy + x^3y^3 \Rightarrow -\dfrac{2}{y^3}\dfrac{dy}{dx} + \dfrac{2x}{y^2} = -2x^3$

Solving the linear equation, we get

$$\frac{1}{y^2}e^{x^2} = -\int 2x^3 \cdot e^{x^2}\,dx = (1-x^2)e^{x^2} + c$$

$y \cdot (0) = 1 \Rightarrow c = 0 \quad \therefore \quad y = \pm\dfrac{1}{\sqrt{1-x^2}}$.

$$\text{Area} = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \pi$$

4. (a) : $AB = 25$, $AC = 15$. If D is the point of intersection of bisector $\angle A$ and the side BC is D, then D divides BC in the ratio $5 : 3$ $\therefore$ $D = \left(-5, -\frac{23}{2}\right)$. The equation joining AD is $11x + 2y + 78 = 0 \Rightarrow c - a = 67$.

5. (d) : If $t = \log_2 x$, the given relation becomes

$$\frac{1}{t^2} - t^2 = \frac{1-t}{1+t} \Rightarrow (t-1)(t^4 + 2t^3 + t^2 + 2t + 1) = 0$$

$$\Rightarrow t = 1,\ t^2 + 2t + 1 + \frac{2}{t} + \frac{1}{t^2} = 0$$

or $\left(t + \frac{1}{t}\right)^2 + 2\left(t + \frac{1}{t}\right) - 1 = 0$

$$\therefore t + \frac{1}{t} = -(\sqrt{2}+1) \in R - (-2, 2)$$

The sum of its roots is $t_1 + t_2 = -(\sqrt{2}+1)$ and product of its roots is $2^{-(\sqrt{2}+1)}$.

6. (a) : $N = 2000^{11} - 2011 = 2^{11}\cdot 10^{33} - 2011$
$= 2048000 \ldots 0000 - 2011 = 2047999 \ldots 97989$
$\therefore$ $S = 2 + 4 + 7 + 7 + 8 + 31 \times 9 = 307$

7. (d) : The tangent to $y = f(x)$ at (x, f) meets the x-axis at $\left(x - \frac{f}{f'}, 0\right)$ ...(i)

The tangent to $y = g(x)$ at (x, g) meets the x-axis at $\left(x - \frac{g}{f'}, 0\right)$ since $g' = f$ ...(ii)

(i) and (ii) are same points. $\therefore$ $g = \dfrac{f^2}{f'}$

Differentiating, $ff'' = (f')^2 \Rightarrow \dfrac{f''}{f'} = \dfrac{f'}{f} \Rightarrow f' = cf, f = Ae^{cx}$

$$f(0) = 1 \Rightarrow f(x) = e^{cx},\ g(x) = \int_{-\infty}^{x} e^{ct}\,dt = \frac{e^{cx}}{c},$$

$$g(0) = \frac{1}{2} \Rightarrow g(x) = \frac{e^{2x}}{2},\ f(x) = e^{2x}$$

$$\lim_{x\to 0}\frac{(f(x))^2 - 1}{x} = \lim_{x\to 0}\frac{e^{4x}-1}{x} = 4$$

8. (d) : Area $= \displaystyle\int_0^1 (g(x) - x)\,dx = \int_0^1\left(\frac{e^{2x}}{2} - x\right)dx = \frac{e^2-3}{4}$

9. (2) : A circle through $(0, -1)$ and $(0, 1)$ is
$x^2 + y^2 - 1 - 2\lambda x = 0$ with centre $(\lambda, 0)$ and radius $\sqrt{\lambda^2+1}$.
It touches the line $y = mx + c$

$$\therefore m\lambda + c = \sqrt{\lambda^2+1}\sqrt{m^2+1}$$

Squaring, $\lambda^2 - 2m\lambda c + m^2 - c^2 + 1 = 0$
If its roots are λ_1, λ_2, then $\lambda_1\lambda_2 = m^2 - c^2 + 1$ ...(i)
The two circles $x^2 + y^2 - 1 - 2\lambda_1 x = 0$ and
$x^2 + y^2 - 1 - 2\lambda_2 x = 0$
cut orthogonally if $\lambda_1\lambda_2 = -1$ ...(ii)
By (i) & (ii), $m^2 - c^2 + 1 = -1 \Rightarrow c^2 - m^2 = 2$

10. (a) : $\begin{vmatrix} 1+\lambda & -\lambda & -2 \\ -\lambda & \lambda-2 & 0 \\ -2 & 0 & 3 \end{vmatrix} = 0 \Rightarrow \lambda = \dfrac{2}{7}$

Pair of lines if $\lambda = \frac{2}{7}$
$\lambda^2 - ab = \lambda^2 - (1 + \lambda)(\lambda - 2) = \lambda + 2$
Parabola if $\lambda = -2$, Ellipse if $\lambda < -2$
Hyperbola if $\lambda > -2$, $\lambda \neq \frac{2}{7}$ (in all three cases)

Solution Sender of Maths Musing

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1. Satya Dev. P (Bangalore)

SET-175

1. Gouri Sankar Adhikary (West Bengal)
2. Suman Bhattacharya (West Bengal)

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